Richard A. Fowell

~~Ralph Hafner~~

~~Feb. 1933~~

PLANE TRIGONOMETRY

PLANE TRIGONOMETRY

with a chapter on
SPHERICAL TRIGONOMETRY

BY
N. J. LENNES
Professor of Mathematics, University of Montana

AND

A. S. MERRILL
Professor of Mathematics, University of Montana

WITH THE EDITORIAL COÖPERATION OF

H. E. SLAUGHT
Professor of Mathematics, University of Chicago

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PLANE TRIGONOMETRY

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EDITOR'S PREFACE

THERE are several reasons why this book is not "just another trigonometry." It has been many years in the making and its essential features have had numerous trials in the classroom. It will be found to be a teachable book, and one whose flexibility will commend itself to the independent and thoughtful teacher.

The editor would call especial attention to the very careful pagination, which is so often neglected in our text-books; to the importance of the cumulative reviews which may be used with telling effect on the students whose habits of thinking have not been well organized; and to the provisions made for the stronger students who will do far more than the rank and file of the class if properly encouraged.

In particular, the editor would commend to teachers the fruitful opportunities open to the leaders of a class in trigonometry for individual reading and study and subsequent reports to the undergraduate Mathematical Club which should be a feature of every college department of mathematics. Topics for such reports will be found in the later chapters of this text and will be suggested at many points in the "Historical Sketch." Such club activities have undergone a rapid growth during the past few years and scores of colleges have found them to be stimulating and effective adjuncts to the regular class work. For further reference to this subject see an address by Professor H. E. Hawkes of Columbia University in the *American Mathematical Monthly* for October, 1918, and the reports and club topics by Professor R. C. Archibald of Brown University in every issue of the *Monthly* for 1918.

H. E. SLAUGHT

UNIVERSITY OF CHICAGO

AUTHORS' PREFACE

In selecting a text for class-room use two questions inevitably arise:

(a) Is the content and arrangement of the book naturally adapted to the type of course that is to be given?
(b) Is the book a good teaching instrument?

These questions were foremost in the authors' minds in making the original plan of this book—the first draft of it was written ten years ago—and have been kept constantly in mind as the various revisions of the manuscript were made. The preface is the proper place to point out the principal features which, in the opinion of the authors, make this book unusually flexible in its adaptability to various types of courses, and which make it a good teaching instrument in the hands of teachers with varied methods of teaching. The authors' aim has been, not to endeavor to impose a stereotyped course and ways of teaching, but to provide a flexible instrument by means of which each teacher may carry out his own purposes in his own way.

General Characteristics.—Effort has been made to attain simplicity and clarity of statement and at the same time rigor and completeness of proof. In some cases where complete proof of a general theorem is necessarily of considerable length and complexity, the proof is made under specified limitations which are removed later. Thus the addition formulas are first proved under the limitation that all angles involved are positive and less than 180°. The formulas thus limited are sufficient for all purposes in solving triangles. In a later chapter these limitations are removed completely. No repetition of proof is involved.

More than ordinary attention has been given to the matter of paging. Pages 64, 65 are typical. On these two pages the formulas for $\sin (A \pm B)$, $\cos (A \pm B)$ are proved. Thus the student has this naturally related material before him without obscuring part of it by turning the page. On pages 72, 73 the law of sines is proved, solving triangles by means of this law is developed, the form of computation given, and exercises provided. On pages

74, 75 the law of tangents and its uses are given, together with appropriate exercises. The study of the table of logarithms of numbers begins on page 34, where reference is made only to logarithms of numbers between 300 and 350 which occur in the page of table printed opposite (page 35). Similar remarks apply to the pages of tables reproduced on pages 37 and 45 and the study of them on the opposite pages and also to the page of natural functions (page 25) and the study and use of it on the page opposite. Only in a very few cases in the latter part of the book is a page turned in the middle of a proof or a discussion. While this arrangement causes a very considerable amount of thought and labor in organizing the material in the first place, and not a little trouble and expense in adjusting the proof in the final paging, these are believed to be justified and even obligatory. Are not the students and teachers in our schools entitled to the same standard of convenience in the instruments they use that is now so rigorously maintained in providing industrial appliances and arrangements?

Arrangement of Exercises and Problems.—Exercises and problems are in nearly all instances provided in two or three groups marked A, B, C. These groups are intended to be of equal difficulty and equally adapted for first use. By this arrangement it is possible to assign different sets to different sections and in successive years. This saves the teacher the trouble of selecting from very long lists that are usually given. These groups of exercises and problems are usually followed by more general lists which on the whole are more difficult and which may be used as extra work for the stronger students. In the case of unusually able classes these may be used to the exclusion of the other groups.

Cumulative Reviews.—On pages 148–152 there is a series of cumulative reviews, the purpose of which is stated briefly on page 147. These reviews are in no sense reviews of the chapters at the end of which it is suggested they be given. For the purpose of organizing them an analysis was made of the essential elements that are brought into the course and of the recurrence of these elements in the course itself. They were then arranged to supplement these recurrences so that the course itself together with the reviews should give each element properly spaced reviews or "relearnings," which we now know are necessary for relatively permanent retention. This arrangement is comparatively novel in text books, and in order not to obtrude it upon teachers who do not wish to make use of it, the reviews are placed in a group near the end of the book, instead of at separate points in the main body of the text.

Provision for Stronger Students Made Possible.—Besides the more general exercises and problems that are given in separate groups in the main body of the book, a series of topics near the end may be used as special assignments for able students who are candidates for the highest grades. Students interested in physics or engineering may read §§183–185 and solve the problems given. Paragraphs 179–182 provide work related to surveying, and 176–178 give some applications of logarithms to problems in business. The final list of miscellaneous theorems and problems beginning on page 164 are on the whole more difficult than the early part of the book and also of more general mathematical interest.

Again Chapters XI, XII ("De Moivre's Theorem," and "Exponential and Hyperbolic Functions") may be assigned for special reports by unusual students. In our own institution such reports have been made before the Mathematics Club, which is an organization of undergraduates interested in mathematics. One of the weaknesses of our college teaching seems to be that the students get but little practice in mastering a subject from the printed page. Certainly the ablest students should be led to do this at an early stage.

Different Courses for Which This Book is Adapted.—The first 90 pages cover that minimum body of material which is included in practically all college courses in trigonometry. An exception is that in some cases Chapter III ("Solution of Right Triangles by Natural Functions") is omitted. This minimum core may be enlarged in a variety of ways by selections from Chapters IX to XIII. In a technical school sections dealing with the variation of the functions, the slide-rule, the problems related to physics, and problems from the final list will naturally be selected. If the interest is primarily in pure mathematics a somewhat different selection will be made, depending largely upon the taste of the instructor. It is believed that the material provided is adequate for all demands which can reasonably be made upon an elementary text such as the present, while at the same time topics of specialized interest have not been introduced into the early parts.

Historical Sketch.—Instead of isolated notes in the body of the text, a historical sketch is given in a separate chapter at the end of the book. This makes possible a continuous story showing something of the development of the subject as a whole, while at the same time making easily accessible information about any particular topic. The sweep of the story, and some of the causal relations between its parts, become more apparent than they can possibly be from isolated notes.

The general editor, Dr. H. E. Slaught, has rendered very much more than perfunctory service. He went over the manuscript before its last revision and made many suggestions. Again, he worked over minutely the final manuscript, and the proof both in galley and in page. The book owes much to his coöperation.

N. J. L.
A. S. M.

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PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

As the name implies (Greek: *trigonon,* triangle, and *metria,* measure), trigonometry had its origin in dealing with problems related to the measuring of the triangle, and this is still one of its chief purposes. Indeed, this is the main topic considered in this text.

1. Problems related to the measuring of the triangle.—In plane geometry it was found that two triangles are equal if

(1) *Two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other;* or if

(2) *Two angles and the included side of one are equal, respectively, to two angles and the included side of the other;* or if

(3) *Three sides of one are equal respectively to the three sides of the other.*

These theorems show that a triangle is fixed if three parts, not all angles, are given.[1]

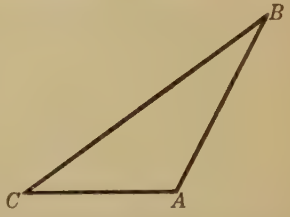
The main problem related to the measurement of the triangle consists in finding formulas by means of which the unknown parts of a triangle may be computed when the known parts are sufficient to determine it.

That is, the problem is to obtain formulas by means of which the unknown parts of a triangle may be found if two sides and the included angle are given, or if two angles and the included side are given, or if the three sides are given.

---

[1] For a necessary modification of this statement see page 80.

2. Practical uses of trigonometry.—If a surveyor wishes to find the distance from an accessible point A to an inaccessible point B, he may lay off a convenient angle at A and measure $\angle A$, the line AC, and $\angle C$. The formulas of trigonometry then enable him to find the distance AB indirectly by computing it. The formulas used by the surveyor in solving this problem are used in practically all indirect measurement of distances.



It is difficult to overestimate the importance of the art of measuring distances indirectly. Without it the surveying and mapping of a mountainous region would be extremely laborious and difficult, and navigation would be nearly impossible. The distances from the earth to the moon, to the sun and the fixed stars would be forever unknown to us. Our ideas about the universe in which we live would in many respects remain as crude and childish as are those of the savage.

Other important practical uses of trigonometry are found in physics and engineering. Some of these are studied briefly in Chapter XIII. See pages 160–164.

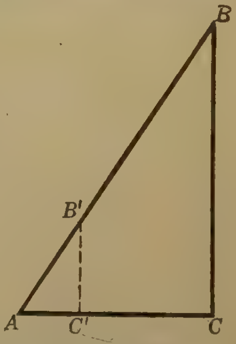
3. Similar triangles used in indirect measurement.—The indirect measurement of distances may be made by means of similar triangles.

Thus, suppose we wish to measure indirectly the distance between two points B and C, at least one of which, as C, is accessible.

Lay off a line CA at right angles to CB. Let B' be a point on the line AB accessible from A. Drop $B'C'$ perpendicular to AC. Then triangles ABC and $AB'C'$ are similar.

Therefore,

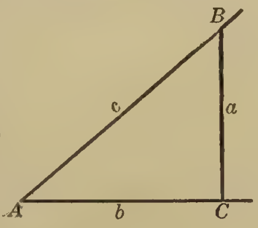
$$\frac{CB}{AC}=\frac{C'B'}{AC'}, \text{ or } CB = AC \times \frac{C'B'}{AC'}.$$



Hence, to find the distance CB, we need to measure AC, AC', and $C'B'$. If this method were to be used in practice, the most difficult and tedious part of the work would be to mark off and measure the segments AC' and $C'B'$. The angle $AC'B'$ would have to be made exactly a right angle and the points C' and B' would have to be placed exactly on the lines AC and AB. In practice this is not so easily done.

In solving this problem by the method of trigonometry the same ratio $C'B'/AC'$ is used, but instead of laying off the segments and measuring them and then computing the ratio, $\angle A$ is measured and the ratio found from tables. Our next step is to study such ratios.[1]

4. The six trigonometric functions for acute angles.—Consider a given acute angle A. On either side of the angle, select any point B, and from B drop a perpendicular BC to the other side. In the right triangle thus formed, designate CB (the side opposite A) by a, AC (adjacent to A but opposite B) by b, and AB, the hypotenuse (opposite C), by c.



There are six ratios of these sides when taken in pairs, namely a/c, b/c, a/b, c/a, c/b, b/a. These are called, respectively, the **sine, cosine, tangent, cosecant, secant,** and **cotangent** of the angle A. The names of these ratios, also called **trigonometric functions,** are usually abbreviated, as shown in the following equations:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c} \qquad \csc A = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} \qquad \cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$$

Notice that these six functions are arranged in pairs so that in each pair either function is the reciprocal of the other. Thus in $\sin A = a/c$ and $\csc A = c/a$, each is the reciprocal of the other. Similarly $\cos A = b/c$ and $\sec A = c/b$ are reciprocals each of the other, as are also $\tan A = a/b$ and $\cot A = b/a$.

A useful form of these definitions is shown in this definition of the sine:

The sine of an acute angle in a right triangle is the side opposite divided by the hypotenuse.

SIGHT WORK

State the definitions of cosine, tangent, cotangent, secant, and cosecant in a form similar to the one given above for the sine.

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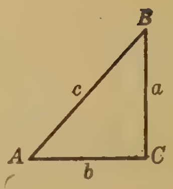
[1] For a review of the geometry involved see page 141.

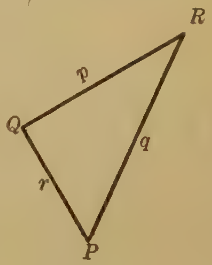
SIGHT WORK

1. If the triangle ABC is right angled at C, then $\sin B$ = side opposite divided by the hypotenuse = b/c. That is, $\sin B = b/c$. (1) In the same figure give $\cos B$ in terms of the sides of the triangle. (2) Also give $\tan B$, $\cot B$, $\sec B$, $\csc B$ in terms of the sides of the triangle.



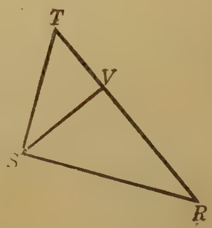
2. The triangle PQR is right angled at Q. Give the six functions of $\angle P$ in terms of p, q, and r.

3. In the same figure give the six functions of $\angle R$ in terms of p, q, and r.



4. In the figure, $\angle RST = 1$ rt $\angle$ and $\angle SVT = 1$ rt $\angle$. Give the six functions of $\angle R$ in terms of RS, ST, and TR. Also give these functions in terms of RS, SV, and VR.

5. In the same figure give the six functions of $\angle T$ in terms of TS, SR, and RT and also in terms of TS, SV, and VT.



6. Give the six functions of $\angle VSR$.

7. Give the six functions of $\angle TSV$.

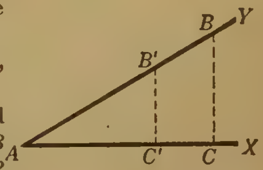
8. On a side AY of the angle A two points B and B' are selected and perpendiculars BC and $B'C'$ are drawn to the other side.

(1) Then are triangles ABC and $AB'C'$ similar? why?

(2) Is the ratio CB/AB equal to $C'B'/AB'$? why?

(3) Do we obtain the same value for $\sin A$ no matter what point on the side AY is selected as the point B? why?

Ask and answer similar questions for $\cos A$, $\tan A$, $\cot A$, $\sec A$, and $\csc A$.



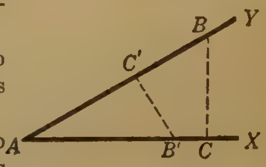
9. In the figure BC is perpendicular to AX and $B'C'$ is perpendicular to AY. Then $\sin A = CB/AB$ and also $\sin A = C'B'/AB'$. Are the ratios equal?

Suggestion: Are triangles ABC and $AB'C'$ similar?

(1) Do we obtain the same value for $\sin A$ no matter on which side of the angle the point B is taken? why?

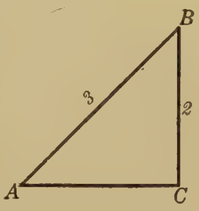
(2) Do we obtain the same value of $\cos A$ no matter on which side of the angle the point B is taken? why?

(3) Ask and answer similar questions for $\tan A$, $\cot A$, $\sec A$, $\csc A$.



10. Explain the statement: The values of the functions of an angle depend upon the angle and not upon the construction made in defining the functions. That is, they are functions of the angle only.

5. Construction of an acute angle having given one of its six functions.—If one of the six functions of an acute angle is given the angle is thereby fixed and we shall now show how to construct such an angle.



Example 1. Construct an angle whose sine is 2/3.

Solution: Construct a right triangle ABC, making the line CB equal to two units and the hypotenuse AB equal to three units.

Then $\sin A = CB/AB = 2/3$.

Example 2. Construct an angle whose cosine is 2/5.

Solution: Construct a right triangle ABC, making the side AC equal to 2 units and the hypotenuse AB equal to 5 units. Then $\cos A = AC/AB = 2/5$.

Example 3. Construct an angle whose tangent is 4/3.

Solution: Construct a right triangle ABC, making CB equal to 4 units and AC equal to 3 units. Then $\tan A = CB/AC = 4/3$.

Example 4. Construct an angle whose cotangent is 3/7.

Solution: Construct a right triangle ABC, making the side AC equal to 3 units and the side CB equal to 7 units. Then $\cot A = AC/CB = 3/7$.

Example 5. Construct an angle whose secant is 6/5.

Solution: Construct a right triangle ABC, making side AC equal to 5 units and the hypotenuse AB equal to 6 units. Then $\sec A = AB/AC = 6/5$.

Example 6. Construct an angle whose cosecant is 7/4.

Solution: Construct a right triangle ABC, making the side CB equal to 4 units and the hypotenuse AB equal to 7 units. Then $\csc A = AB/CB = 7/4$.

EXERCISES

1. Construct an angle whose sine is 3/7. Is it possible to construct an angle whose sine is 4/3? why?

2. Construct an angle whose cosine is 3/4. Is it possible to construct an angle whose cosine is 4/3? why?

3. Construct an angle whose tangent is 3/5. Is this angle greater than or less than 45°? why? Is an angle whose tangent is 5/3 greater or less than 45°? why?

4. Construct an angle whose cotangent is 4/3. Is this angle greater or less than 45°? why?

5. Construct an angle whose secant is 7/3. Is it possible to construct an angle whose secant is less than 1? why?

6. Construct an angle whose cosecant is 3/2. Is it possible to construct an angle whose cosecant is less than 1? why?

7. If the sine of an angle is 3/5 what is its cosecant?

8. If the secant of an angle is 3/2 what is its cosine?

9. If the tangent of an angle is 5/3 what is its cotangent?

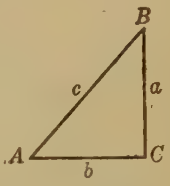
10. If $\tan A = 3/4$ and $\tan B = 5/3$, which is greater, A or B?

11. If $\cot A = 3/4$ and $\cot B = 5/3$, which is greater, A or B?

6. Complementary angles.—If the sum of two angles equals one right angle, or 90°, the angles are said to be **complementary.**

Thus, the two acute angles of a right triangle are complementary.

7. Functions of complementary angles.—In the triangle ABC, $\sin A = a/c$ while $\cos B = a/c$. Similarly, $\tan A = a/b$ while $\cot B = a/b$, and $\sec A = c/b$ while $\csc B = c/b$. Angle A and angle B are complementary. This gives rise to the following rules:



If $\angle A$ and $\angle B$ are complementary acute $\angle$s, then $\sin A = \cos B$; $\tan A = \cot B$; $\sec A = \csc B$. This may also be stated as follows:

(1) *The sine of an angle is equal to the cosine of its complement.*

(2) *The tangent of an angle is equal to the cotangent of its complement.*

(3) *The secant of an angle is equal to the cosecant of its complement.*

Since A and $90° - A$ are complementary angles, this rule may be stated as follows:

$$\sin A = \cos (90° - A); \quad \cos A = \sin (90° - A) \qquad [1]$$
$$\tan A = \cot (90° - A), \quad \cot A = \tan (90° - A) \qquad [2]$$
$$\sec A = \csc (90° - A), \quad \csc A = \sec (90° - A) \qquad [3]$$

8. Cofunctions of an angle.—The sine and the cosine of an angle are called **cofunctions** of the angle. Similarly, the tangent and cotangent of an angle are cofunctions of the angle, as are also the secant and the cosecant. The rule of § 7 may now be expressed briefly by saying:

Any function of an angle equals the cofunction of the complement of that angle.

This is a convenient rule and is easily remembered by associating the words "complement" and "cofunction."

We may also state the converse of this proposition, namely:

Two acute angles A and B are complementary if,
(1) $\sin A = \cos B$, (2) $\tan A = \cot B$, (3) $\sec A = \csc B$.

That is, *if the cofunctions of two acute angles are equal, the angles are complementary.*

This rule does not hold unless the angles are acute.

SIGHT WORK

In the exercises on this page it is assumed that all angles are acute and that hence the rules obtained in §§ 7, 8 apply.

1. If the sum of two angles is 90°, what is the relation between the sine of one of them and the cosine of the other?

2. If the tangent of one of two angles is equal to the cotangent of the other, what is the relation between these angles?

3. If the sum of two angles is 90°, what is the relation between the secant of one of them and the cosecant of the other?

4. Find the value of x if $\sin (2x - 20°) = \cos (3x + 10°)$.

Suggestion. Since $\sin (2x - 20°) = \cos (3x + 10°)$ it follows that $2x - 20° + 3x + 10° = 90°$.

5. Find A if $\sin A = \cos 3A$.

6. Find A if $\sin \frac{2}{3} A = \cos 2A$.

7. Find A if $\sec 4A = \csc 3A$.

8. Find x if $\tan (5x + 15°) = \cot (7x - 20°)$.

9. Find x if $\sec (2x + 40°) = \csc (5x - 30°)$.

10. Find A and B if $\sin A = \cos B$ and if $A - B = 20°$.

11. Find A and B if $\sin A = \cos 2B$ and if $B + 10° = A$.

12. Find A and B if $\tan 3A = \cot B$ and $A - B = 10°$.

13. Find A and B if $\cot (A + 10°) = \tan B$ and $A - B = 40°$.

14. Find A and B if $\tan (B - 20°) = \cot 5A$ and $4A - B = 25°$.

15. Find x and y if $\sec (x + 18°) = \csc (y + 40°)$ and $5x - 2y = 20°$.

16. Find x and y if $\csc (x - 50°) = \sec (y - 10°)$ and $x - y = 20°$.

17. Find x and y if $\sec (4x - 24°) = \csc (x - y + 18°)$ and $x + y = 48°$.

9. Functions of 45°.—If in the right triangle ABC, $\angle A = 45°$, then $\angle B = 45°$ and $AC = CB$. If $CB = a$, then $\overline{AB}^2 = a^2 + a^2$ and $AB = a\sqrt{2}$.

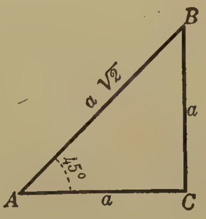
Hence,

$$\sin 45° = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\cos 45° = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\tan 45° = \frac{a}{a} = 1, \ \cot 45° = \frac{a}{a} = 1,$$

$$\sec 45° = \frac{a\sqrt{2}}{a} = \sqrt{2}, \ \csc 45° = \frac{a\sqrt{2}}{a} = \sqrt{2}.$$



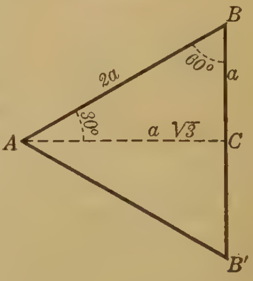
SIGHT WORK

1. If you know that $\sin 45° = \frac{1}{2}\sqrt{2}$, how can you find the value of $\csc 45°$? What rule do you use?

2. If you know that $\cos 45° = \frac{1}{2}\sqrt{2}$, how can you find the value of $\sec 45°$? What general rule do you use?

3. If you know that $\tan 45° = 1$, how can you find the value of $\cot 45°$?

10. Functions of 30°, 60°.—A right triangle having one angle of 30° possesses certain special relations among its sides which make it possible to find directly the functions of 30° and 60°.



In the equilateral triangle ABB', draw AC bisecting $\angle A$. Then AC is the perpendicular bisector of $B'B$. Since an equilateral triangle is also equiangular, it follows that $\angle B$ equals 60° and $\angle CAB$ equals 30°. Further, the side CB equals $\frac{1}{2}$ of AB, since CB is $\frac{1}{2}$ of $B'B$.

If $CB = a$, then $AB = 2a$ and $AC = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}$.
Hence,

$$\sin CAB = \sin 30° = \frac{a}{2a} = \frac{1}{2},$$

$$\cos 30° = \frac{a\sqrt{3}}{2a} = \frac{1}{2}\sqrt{3},$$

$$\tan 30° = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

In the same manner we find,

$$\cot 30° = \sqrt{3},\ \sec 30° = \frac{2}{3}\sqrt{3},\ \text{and}\ \csc 30° = 2.$$

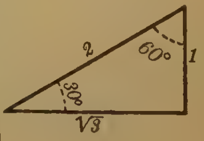
Since angles of 30° and 60° are complementary, it follows that,

$$\sin 60° = \frac{1}{2}\sqrt{3},\ \cos 60° = \frac{1}{2},\ \tan 60° = \sqrt{3}.$$

$$\cot 60° = \frac{1}{3}\sqrt{3},\ \sec 60° = 2,\ \csc 60° = \frac{2}{3}\sqrt{3}.$$

11. Table of functions of 30°, 45°, 60°.—The following table will help in remembering sine, cosine, and tangent of 30°, 45°, 60°.

Func.	30°	45°	60°
Sine	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
Cosine	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
Tangent	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$



The sines may be remembered by noticing that for 30°, 45°, 60° they are in order 1/2 of $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$; or 1/2, $\sqrt{2}/2$, $\sqrt{3}/2$. The cosines of these angles are the same numbers with the order reversed. The tangent may be remembered by noticing that $\sqrt{3} = \tan 60°$, while $\frac{1}{3}\sqrt{3} = \tan 30°$.

These functions of 30° and 60° may also be read directly from the figure given to the right of the table.

EXERCISES

Using the principle of § 8, express each of the following as a function of an angle less than 45°:

1. sin 50°.
2. cos 60°.
3. tan 60°.
4. sin 70°.
5. cos 80°.
6. cot 70°.
7. sec 65°.
8. csc 75°.
9. sin 80°.
10. tan 55°.
11. cot 75°.
12. cos 48°.
13. cot 54°.
14. sec 63°.
15. sin 71°.
16. cos 84°.

Express as functions of angles between 45° and 90°:

17. sin 15°.
18. cos 34°.
19. tan 10°.
20. cot 24°.
21. sec 41°.
22. cos 39°.
23. csc 16°.
24. tan 32°.
25. sin 28°.
26. cos 13°.
27. tan 36°.
28. cot 21°.
29. sec 7°.
30. cot 14°.
31. csc 18°.
32. cos 37°.

Express as functions of the angle A:

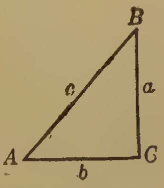
33. $\sin (90^\circ - A)$.
34. $\cos (90^\circ - A)$.
35. $\tan (90^\circ - A)$.
36. $\cot (90^\circ - A)$.
37. $\sec (90^\circ - A)$.
38. $\csc (90^\circ - A)$.

Express as functions of $90^\circ - A$:

39. $\sin A$.
40. $\tan A$.
41. $\cot A$.
42. $\cos A$.
43. $\csc A$.
44. $\sec A$.

Using § 8 and the suggestion under Ex. 4, p. 7, find $\angle A$, given the following relations:

45. $\sin (2A + 10^\circ) = \cos 4A$.
46. $\sin 5A = \cos (3A - 15^\circ)$.
47. $\tan 3A = \cot (2A - 40^\circ)$.
48. $\cot (6A - 40^\circ) = \tan 2A$.
49. $\sec 3A = \csc (5A + 10^\circ)$.
50. $\csc 4A = \sec (7A + 15^\circ)$.



By the Pythagorean proposition $a^2 + b^2 = c^2$ if ABC is a right triangle. Hence it is possible to construct a right triangle whose sides are 3, 4, 5. (Notice that $3^2 + 4^2 = 5^2$.)

51. Find the functions of angle A in the triangle ABC if its sides are $a = 4$, $b = 3$, and $c = 5$. Also find the functions of the angle B. Which is greater, $\sin A$ or $\tan A$? Which is greater, $\cos A$ or $\cot A$?

52. Find the functions of angle A if $a = 5$, $b = 12$, and $c = 13$. Also find the functions of angle B.

Find the functions of the acute angles of a right triangle if its sides are:

53. 8, 15, 17.
54. 10, 24, 26.
55. 12, 35, 37.
56. 3.9, 8, 8.9.
57. 4, 5, $\sqrt{41}$.
58. 3, 8, $\sqrt{73}$.

EXERCISES

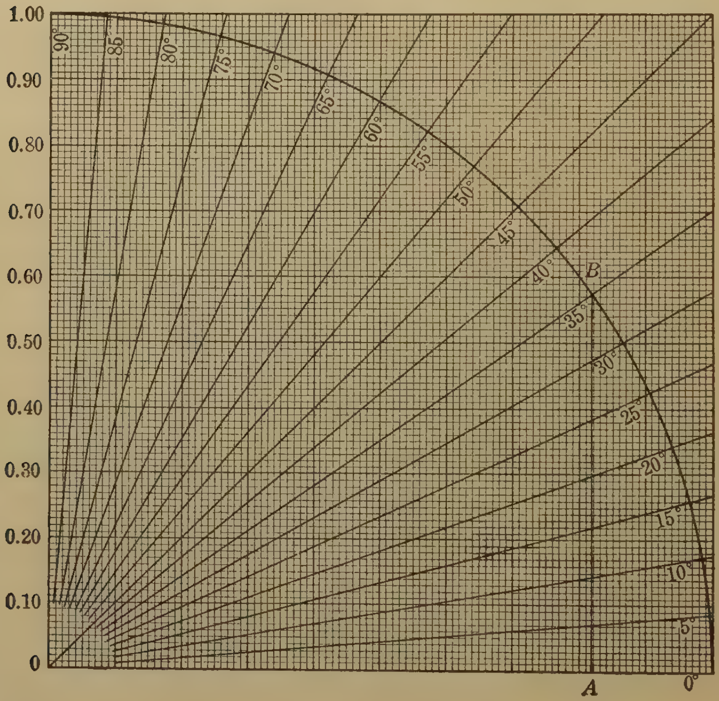
Problems in construction and measurement. (This page may be omitted if the teacher does not wish to include work in actual measurement by the pupil. If this work is included § 168 should be read now.)

1. Construct an angle whose sine is 3/4 and find by measurement with a protractor the number of degrees in the angle.

2. Construct an angle whose cosine is 4/7 and find by measurement the number of degrees in the angle.

3. Construct an angle whose tangent is 4/5 and find by measurement the number of degrees in the angle.

4. Construct a right triangle having an angle of 35°. Measure the sides of the triangle and compute the functions of 35°.



5. On a large sheet of ruled paper construct one-fourth of a circle, using 100 small spaces as a radius. Construct angles of 5°, 10°, 15°, 20°, etc., up to 85°, and read off the sines and cosines. Since the radius of the circle is 100 small units, the vertical and horizontal distances from the axes to the points where the sides of the angles cut the circle will give the sines and cosines, respectively. Compare the values obtained by measurement with the more accurate values given on page 11.

Func.	10°	20°	30°	40°	50°	60°	70°	80°
Sin	.174	.342	.500	.643	.766	.866	.940	.985
Cos	.985	.940	.866	.766	.643	.500	.342	.174
Tan	.176	.364	.577	.839	1.192	1.732	2.747	5.671

12. Algebraic Operations with Trigonometric Expressions.—From the definitions of the various functions of an acute angle it is evident that each function of a given angle is a number. Thus $\sin A$ and $\cos A$ stand for numbers, just as in algebra the letters a and x stand for numbers. It follows that algebraic operations, such as addition, subtraction, multiplication, division, involution, and evolution, may be performed upon these functions. In general, these are done in the same manner as in algebra. Certain forms, however, should be noted.

The product of $\sin A$ and $\cos A$ is $\sin A \cos A$, and the sum of $\sin A$ and $\sin B$ is $\sin A + \sin B$. The product of $\sin A$ multiplied by itself is $(\sin A)^2$. It is usual, however, to write instead $\sin^2 A$, and this expression is read "sine squared of A," or more briefly "sine square A." Similarly we have $\cos^2 A$, $\tan^2 A$, $\sin^2 A$, etc.

The expression $\sin (A + B)$ stands for the sine of the angle obtained by adding the angles A and B. It is natural to inquire whether $\sin A + \sin B$ is equal to $\sin (A + B)$. We certainly have no reason to believe that such is the case, and will find later that in general the two expressions are not equal.

The expression $\sin 4A$ stands for the sine of the angle $4A$. Again, $\sin 4A$ is not in general equal to $4 \sin A$.

A few problems will illustrate the use of trigonometric expressions in algebraic processes.

Example 1. Find the product of $(\cos A + \sin A)(\cos A - \sin A)$.

Solution: Since this is the product of the sum and difference of two terms, the result is the difference of their squares. That is,

$$(\cos A + \sin A)(\cos A - \sin A) = \cos^2 A - \sin^2 A.$$

Example 2. Perform the indicated addition:

$$\frac{\sin A}{\cos B} + \frac{\sin B}{\cos A}.$$

Solution: Evidently $\cos B \cos A$ is the common denominator. Then

$$\frac{\sin A}{\cos B} + \frac{\sin B}{\cos A} = \frac{\sin A \cos A}{\cos B \cos A} + \frac{\sin B \cos B}{\cos B \cos A}$$

$$= \frac{\sin A \cos A + \sin B \cos B}{\cos B \cos A}.$$

Example 3. Factor $\tan A \sin A - \sin^2 A$.

Solution: $\sin A$ is a factor common to both terms.

Hence

$$\tan A \sin A - \sin^2 A = \sin A\,(\tan A - \sin A).$$

EXERCISES

GROUP A

Perform the following indicated operations:

1. $(\sin A + \cos B)^2 = ?$
2. $\dfrac{\tan A}{\cos B} - \dfrac{\sin B}{\tan B \cos B} = ?$
3. $\sqrt{\sin^4 A \cos^2 B} = ?$
4. $1 + \dfrac{\cos^2 A}{\sin^2 A} = ?$
5. $\left(\sin A + \dfrac{\sqrt{2}}{2}\right)\left(\sin A - \dfrac{\sqrt{2}}{2}\right) = ?$

Factor the following expressions:

6. $\cos^2 A - \sin A \cos A$.
7. $\cos^4 A - \cos^2 B$.
8. $\sin^4 A - \cos^4 A$.
9. $\tan^2 A - 2 \tan A \cot A + \cot^2 A$.
10. $\sin^2 A - \sin^2 A \cos^2 A$.
11. $\sin^2 A - \sin A \sec A - 6 \sec^2 A$.

Simplify the following expressions:

12. $\dfrac{\dfrac{\sin A}{\cos A}}{\dfrac{\cos A}{\sin^2 A}}$.
13. $\dfrac{\dfrac{\sin A}{\cos B} + \dfrac{\cos A}{\sin B}}{\sin B \cos B}$.

GROUP B

1. $(\sin A + \sin B)(\cos A + \sin B) = ?$
2. $\sqrt{\tan^2 B \cos^6 A} = ?$
3. $\dfrac{\cos A}{\tan B} + \dfrac{\cot B}{\cos A \tan B} = ?$
4. $\dfrac{\sin^2 A}{\cos^2 A} + 1 = ?$
5. $\left(\tan A + \dfrac{1}{3}\sqrt{3}\right)\left(\tan A - \dfrac{1}{3}\sqrt{3}\right) = ?$.

Factor the following expressions:

6. $\sin^2 A + \sin A \tan A$.
7. $\sin^2 A \cos A - \sin A \cos^2 A$.
8. $\sin^2 A \cos^3 A + \sin^3 A \cos^2 A$.
9. $\sin^2 A + 2 \sin A \cos A + \cos^2 A$.
10. $\tan^3 A \cos^2 A - \tan^2 A \cos^3 A + \tan A \cos A$.
11. $\cos^2 A + \sin^2 A \cos A - 2 \sin^4 A$.

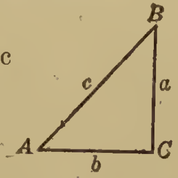
Simplify:

12. $\dfrac{1 + \dfrac{\sin^2 A}{\cos^2 A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}}$
13. $\dfrac{\dfrac{\tan A}{\cot A} - \dfrac{\cot A}{\tan A}}{\dfrac{\tan A}{\cot A} + \dfrac{\cot A}{\tan A}}$.

CHAPTER II

RELATIONS BETWEEN THE FUNCTIONS OF AN ANGLE

THERE are certain relations between the functions of an angle which hold entirely independently of the size of the angle. Such relations are called *trigonometric identities.* We shall find in Chapter VI that these identities hold for functions of any angle whatever. In this chapter they are considered for functions of acute angles only.



13. Reciprocal relations.—The six trigonometric functions are reciprocals in pairs.

Thus, from $\sin A = a/c$ and $\csc A = c/a$ it follows that

$$\sin A = \frac{1}{\csc A}, \qquad \csc A = \frac{1}{\sin A} \qquad [4]$$

Again, from $\cos A = b/c$ and $\sec A = c/b$ it follows that

$$\cos A = \frac{1}{\sec A}, \qquad \sec A = \frac{1}{\cos A} \qquad [5]$$

Since $\tan A = a/b$ and $\cot A = b/a$, it follows that

$$\tan A = \frac{1}{\cot A}, \qquad \cot A = \frac{1}{\tan A} \qquad [6]$$

These identities may also be stated in the following form:

$$\sin A \csc A = 1, \cos A \sec A = 1, \tan A \cot A = 1.$$

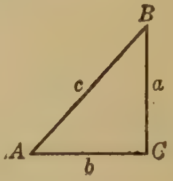
14. Trigonometric functions expressed in terms of sines and cosines. In § 13 we have seen that the secant may be expressed in terms of cosine and cosecant in terms of sine. The tangent and cotangent may be expressed in terms of sine and cosine. Thus,

$$\tan A = a/b = \frac{a/c}{b/c} = \frac{\sin A}{\cos A} \qquad [7]$$

$$\cot A = b/a = \frac{b/c}{a/c} = \frac{\cos A}{\sin A} \qquad [8]$$

It follows that in any expression containing trigonometric functions these may all be expressed in terms of sines and cosines.

15. Relations derived from the Pythagorean proposition.—Since in the figure, c is the hypotenuse of a right triangle, and b and a are the other two sides, we have



$$a^2 + b^2 = c^2.$$

Dividing these equals successively by c^2, b^2, and a^2, we obtain the following results:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}, \text{ or } \sin^2 A + \cos^2 A = 1 \qquad [9]$$

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}, \text{ or } \tan^2 A + 1 = \sec^2 A \qquad [10]$$

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}, \text{ or } \cot^2 A + 1 = \csc^2 A \qquad [11]$$

16. List of formulas.—The following list contains the most important formulas on the relations among the functions of an angle:

(1) $\sin A = \dfrac{1}{\csc A}$ (2) $\cos A = \dfrac{1}{\sec A}$ (3) $\tan A = \dfrac{1}{\cot A}$

(4) $\tan A = \dfrac{\sin A}{\cos A}$ (5) $\cot A = \dfrac{\cos A}{\sin A}$ (6) $\sin^2 A + \cos^2 A = 1$

(7) $\tan^2 A + 1 = \sec^2 A$ (8) $\cot^2 A + 1 = \csc^2 A$.

SIGHT WORK

1. From $\sin^2 A + \cos^2 A = 1$ show that $\sin^2 A = 1 - \cos^2 A$ and hence that $\sin A = \sqrt{1 - \cos^2 A}$.

2. From $\sin^2 A + \cos^2 A = 1$ show that $\cos A = \sqrt{1 - \sin^2 A}$

Remark. The two formulas $\sin A = \sqrt{1 - \cos^2 A}$ and $\cos A = \sqrt{1 - \sin^2 A}$ are used very frequently in solving problems.

3. From $\tan^2 A + 1 = \sec^2 A$ show that $\tan A = \sqrt{\sec^2 A - 1}$.

4. From $\cot^2 A + 1 = \csc^2 A$ show that $\cot A = \sqrt{\csc^2 A - 1}$.

5. From $\tan A = \sin A/\cos A$ and $\cot A = \cos A/\sin A$ show that $\tan A \cot A = 1$.

6. What formula is represented by the statement "the sine of an angle is equal to the reciprocal of its cosecant"?

7. What formula is represented by the statement, "the cosine of an angle is equal to the reciprocal of its secant?"

8. State in words each of the formulas (3) — (8) above.

17. Finding all the functions of an angle when one function is given.—When one function of an acute angle is given, the other functions may be found by constructing a right triangle one of whose acute angles has the given function. The other functions may then be read directly from the triangle.

Example 1. Find the functions of an angle whose sine is 2/3.

Solution: Construct a right triangle ABC such that CB is two units long and AB is three units.

Then $\sin A = CB/AB = 2/3$.

From the theorem of Pythagoras we have,

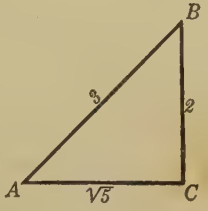
$$3^2 = 2^2 + \overline{AC}^2 \text{ or } \overline{AC}^2 = 3^2 - 2^2 = 5.$$

Hence,

$$AC = \sqrt{5}.$$

We now read at once,

$$\cos A = \sqrt{5}/3, \tan A = 2/\sqrt{5}, \cot A = \sqrt{5}/2,$$
$$\sec A = 3/\sqrt{5}, \text{ and } \csc A = 3/2.$$



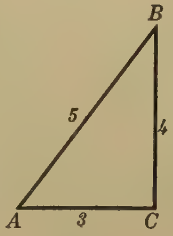
From these values verify (1) that $\sin A = 1/\csc A$, (2) $\cos A = 1/\sec A$, (3) $\tan A = 1/\cot A$, (4) $\tan A = \sin A/\cos A$, (5) $\cot A = \cos A/\sin A$, (6) $\sin^2 A + \cos^2 A = 1$, (7) $\tan^2 A + 1 = \sec^2 A$, (8) $\cot^2 A + 1 = \csc^2 A$.

Example 2. Find the functions of an acute angle whose secant is 5/3.

Solution: Construct a right triangle ABC with AB five units and AC three units. Then CB is four units.

$$\text{Then, } \sec A = 5/3, \sin A = 4/5, \cos A = 3/5,$$
$$\tan A = 4/3, \cot A = 3/4, \csc A = 5/4.$$

From these values of the functions verify as in Example 1 formulas (1)-(8) on page 14.



EXERCISES

1. Construct an angle whose cosine is 3/7, and find all its other functions. From these values of the functions verify formulas (1)–(8) on page 14.

2. Construct an angle whose tangent is 5/7 and find all its other functions. Verify formulas as in problem 1.

3. Construct an angle whose cotangent is 7/8 and find its other functions. Verify formulas as above.

4. Construct an angle whose cosecant is 8/5 and find its other functions. Verify formulas as above.

5. Draw an angle whose sine is a/c and find its other functions in terms of a and c. Using these values of the functions, verify formulas as in the preceding examples.

18. Expressing trigonometric functions in terms of one function.—All functions of an angle may be expressed in terms of any one function. For this purpose the eight formulas collected in § 16 are used. The general method is shown in the following examples:

Example 1. Express all trigonometric functions in terms of the sine.

Solution: From $\sin^2 A + \cos^2 A = 1$ we have $\cos^2 A = 1 - \sin^2 A$ or $\cos A = \sqrt{1 - \sin^2 A}$.

Also $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\sin A}{\sqrt{1 - \sin^2 A}}$,

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

Example 2. Express all trigonometric functions in terms of the tangent.

Solution: Since $\tan A \cot A = 1$, $\cot A = \dfrac{1}{\tan A}$.

Since $\sec^2 A = 1 + \tan^2 A$, $\sec A = \sqrt{1 + \tan^2 A}$,

and $\cos A = \dfrac{1}{\sec A} = \dfrac{1}{\sqrt{1 + \tan^2 A}}$.

Now, $\csc^2 A = 1 + \cot^2 A = 1 + \dfrac{1}{\tan^2 A} = \dfrac{\tan^2 A + 1}{\tan^2 A}$.

Hence, $\csc A = \dfrac{\sqrt{1 + \tan^2 A}}{\tan A}$ and $\sin A = \dfrac{\tan A}{\sqrt{1 + \tan^2 A}}$.

EXERCISES

Derive the values given in the following table from the fundamental identities of § 16.

$\sin A =$	$\sin A$	$\sqrt{1-\cos^2 A}$	$\dfrac{\tan A}{\sqrt{1+\tan^2 A}}$	$\dfrac{1}{\sqrt{1+\cot^2 A}}$	$\dfrac{\sqrt{\sec^2 A-1}}{\sec A}$	$\dfrac{1}{\csc A}$
$\cos A =$	$\sqrt{1-\sin^2 A}$	$\cos A$	$\dfrac{1}{\sqrt{1+\tan^2 A}}$	$\dfrac{\cot A}{\sqrt{1+\cot^2 A}}$	$\dfrac{1}{\sec A}$	$\dfrac{\sqrt{\csc^2 A-1}}{\csc A}$
$\tan A =$	$\dfrac{\sin A}{\sqrt{1-\sin^2 A}}$	$\dfrac{\sqrt{1-\cos^2 A}}{\cos A}$	$\tan A$	$\dfrac{1}{\cot A}$	$\sqrt{\sec^2 A-1}$	$\dfrac{1}{\sqrt{\csc^2 A-1}}$
$\cot A =$	$\dfrac{\sqrt{1-\sin^2 A}}{\sin A}$	$\dfrac{\cos A}{\sqrt{1-\cos^2 A}}$	$\dfrac{1}{\tan A}$	$\cot A$	$\dfrac{1}{\sqrt{\sec^2 A-1}}$	$\sqrt{\csc^2 A-1}$
$\sec A =$	$\dfrac{1}{\sqrt{1-\sin^2 A}}$	$\dfrac{1}{\cos A}$	$\sqrt{1+\tan^2 A}$	$\dfrac{\sqrt{1+\cot^2 A}}{\cot A}$	$\sec A$	$\dfrac{\csc A}{\sqrt{\csc^2 A-1}}$
$\csc A =$	$\dfrac{1}{\sin A}$	$\dfrac{1}{\sqrt{1-\cos^2 A}}$	$\dfrac{\sqrt{1+\tan^2 A}}{\tan A}$	$\sqrt{1+\cot^2 A}$	$\dfrac{\sec A}{\sqrt{\sec^2 A-1}}$	$\csc A$

19. Proving trigonometric identities.—By means of the eight fundamental identities given on page 14, a large number of identities may be proved. The method of making such proofs is best understood from examples.

Example 1. Prove that $\sin^2 A \cos^2 A + \cos^4 A = 1 - \sin^2 A$ is an identity.

Proof: $\sin^2 A \cos^2 A + \cos^4 A = \cos^2 A (\sin^2 A + \cos^2 A) = \cos^2 A,$

$$\text{since} \quad \sin^2 A + \cos^2 A = 1.$$

From this last identity we also have $\cos^2 A = 1 - \sin^2 A$.

Example 2. Prove that $\dfrac{1}{\tan A + \cot A} = \sin A \cos A$ is an identity. (1)

Proof: Clearing (1) of fractions,

$$1 = \tan A \sin A \cos A + \cot A \sin A \cos A. \tag{2}$$

Substituting $\tan A = \dfrac{\sin A}{\cos A}$ and $\cot A = \dfrac{\cos A}{\sin A}$ in (2),

$$1 = \frac{\sin A}{\cos A} \sin A \cos A + \frac{\cos A}{\sin A} \sin A \cos A, \tag{3}$$

or

$$1 = \sin^2 A + \cos^2 A, \tag{4}$$

which is a known identity.

In this proof each of the equalities (2), (3), (4) is obtained from the one preceding it by an algebraic transformation or by substitution from one of the fundamental identities. Hence we have proved that (2) follows from (1), (3) follows from (2), and (4) follows from (3). But (4) is known to be an identity. We now notice that each of these steps may be reversed. That is, (3) may be proved from (4), (2) from (3), and (1) from (2). Hence (1) has been proved to be a true identity. The general method is to begin with the proposed identity and obtain from it a known identity. This proves the proposed identity, since each step is reversible. For an exception to this general rule see pages 140, 141.

SIGHT WORK

Prove that each of the following is an identity:

1. $\sin A \cot A = \cos A$.
2. $\cos A \tan A = \sin A$.
3. $\sec A \cot A = \csc A$.
4. $\csc A \tan A = \sec A$.
5. $\sec A \sin A = \tan A$.
6. $\cos A \csc A = \cot A$.
7. $\sin A / \csc A = \sin^2 A$.
8. $\cos A / \sec A = \cos^2 A$.
9. $\sin^2 A / \csc^2 A = \sin^4 A$.
10. $\cos^2 A / \sec^2 A = \cos^4 A$.
11. $\sin^2 A / \cot^2 A = \sin^4 A \sec^2 A$.
12. $\cot A / \csc A = \cos A$.
13. $\sin A \sec A \cot A = 1$.
14. $\cos A \csc A \tan A = 1$.
15. $\sin^2 A / \cos^2 A = \sec^2 A - 1$.
16. $\cos^2 A / \sin^2 A = \csc^2 A - 1$.
17. $\sin^2 A \cot A = \sin A \cos A$.
18. $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.
19. $\sin^2 A - \cos^2 A = 1 - 2 \cos^2 A$.
20. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.

WRITTEN EXERCISES (A)

Prove that each of the following is an identity:

1. $\cot A \cos A + \sin A = \csc A.$

2. $\cot A + \tan A = \sec A \csc A.$

3. $\tan^2 A - \sin^2 A = \sin^2 A \tan^2 A.$

4. $\tan^2 A - \cot^2 A = \sec^2 A - \csc^2 A.$

5. $\csc^4 A - \cot^4 A = \csc^2 A + \cot^2 A.$

6. $\sec^2 A + \csc^2 A = \tan^2 A + \cot^2 A + 2.$

7. $\dfrac{\cos A + \sin A}{\sec A + \csc A} = \cos A \sin A.$

8. $\dfrac{\cot A \cos A}{\cot A - \cos A} = \dfrac{\cos A}{1 - \sin A}.$

9. $\dfrac{\sin A + \tan A}{\csc A + \cot A} = \sin A \tan A.$

10. $\dfrac{\tan A - \sin A}{\sin^3 A} = \dfrac{1 - \cos A}{\sin^2 A \cos A}.$

11. $\dfrac{\cot A + 1}{\sin A + \cos A} = \csc A.$

12. $\dfrac{1}{1 + \sin A} + \dfrac{1}{1 - \sin A} = 2 \sec^2 A.$

13. $\dfrac{\sec A + 1}{\tan A} = \dfrac{\tan A}{\sec A - 1}.$

14. $\dfrac{1}{\sec A + \tan A} = \sec A - \tan A.$

15. $(\tan A + \cot A) \sin A \cos A = 1.$

16. $\cot A - \sec A \csc A (1 - 2 \sin^2 A) = \tan A.$

WRITTEN EXERCISES (B)

Prove that each of the following is an identity:

1. $1 + \cot^2 A = \cot^2 A \sec^2 A.$

2. $\tan A \sin A + \cos A = \sec A.$

3. $\sin^2 A \cos^2 A + \cos^4 A = 1 - \sin^2 A$

4. $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$

5. $\dfrac{\cos A + \cot A}{\sec A + \tan A} = \cos A \cot A$

6. $\tan^4 A + \tan^2 A = \sec^4 A - \sec^2 A$

7. $\dfrac{\cos^2 A}{1 + \sin A} = 1 - \sin A$

8. $\dfrac{\cot A - \cos A}{\cot A} = 1 - \sin A$

9. $\dfrac{\csc A}{\cot A + \tan A} = \cos A$

10. $\dfrac{\tan A - \cot A}{\tan A + \cot A} = 2 \sin^2 A - 1$

11. $\dfrac{\sec^2 A + \csc^2 A}{\sec A \csc A} = \tan A + \cot A$

12. $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}$

13. $\dfrac{\sec A}{\cot A + \tan A} = \sin A$

14. $\sec^4 A - 1 = 2 \tan^2 A + \tan^4 A$

15. $(\tan A - \cot A) \sin A \cos A = \sin^2 A - \cos^2 A$

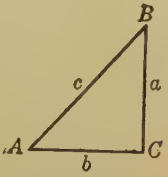
16. If $x = r \cdot \cos A$ and $y = r \cdot \sin A$, prove $x^2 + y^2 = r^2$

CHAPTER III

SOLUTION OF RIGHT TRIANGLES BY MEANS OF NATURAL FUNCTIONS

We know from geometry that a right triangle is determined by any two elements besides the right angle, provided at least one of them is a side.

Thus, the right triangle ABC with sides a, b, c is determined by each of the pairs of elements: a, b; a, c; b, c; $a, \angle A$; $b, \angle A$; $c, \angle A$; etc. (Notice that the triangle is not determined by $\angle A$, $\angle B$.) We shall now obtain formulas by means of which the remaining parts of a right triangle may be computed when enough parts are given to determine it.



20. Formulas derived from the definitions of the trigonometric functions.—In each of the equations giving the definitions of trigonometric functions, three quantities are involved. Hence, each of these equations may be solved for one of these quantities in terms of the other two.

That is, $\sin A = a/c$ may be solved for a or for c, giving, $a = c \sin A$ and $c = a/\sin A$. Hence, if c and $\sin A$ are given, a may be found, and if a and $\sin A$ are given, c may be found.

It is by means of such formulas and a table of trigonometric functions (see page 25) that the unknown parts of the triangles are found.

The equation $a = c \sin A$ may be stated in the form of a rule as follows:

To find the side opposite an acute angle A in a right triangle, multiply the hypotenuse by $\sin A$.

SIGHT WORK

1. State $c = a/\sin A$ in the form of a rule.

2. From the equation $\cos A = b/c$ give a rule for finding b when $\cos A$ and c are known, and also a rule for finding c when $\cos A$ and b are known.

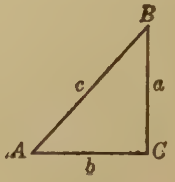
For each of the following equations give a rule for finding each of the three elements involved when the other two are given:

3. $\tan A = \frac{a}{b}$ **4.** $\cot A = \frac{b}{a}$ **5.** $\sec A = \frac{c}{b}$ **6.** $\csc A = \frac{c}{a}$

21. Selection of formulas.—The work of computing may be reduced to a minimum by the proper selection of formulas. Unless logarithms are used (see page 29), care should be taken to obtain formulas which require multiplication rather than division, since the latter operation is more laborious. This is illustrated in the paragraphs that follow.

All formulas used in finding unknown parts of a right triangle are obtained from the fundamental relations:

(1) $\sin A = a/c$, (2) $\cos A = b/c$,
(3) $\tan A = a/b$, (4) $\csc A = c/a$,
(5) $\sec A = c/b$, (6) $\cot A = b/a$.



22. Formulas giving side opposite.—The side a is contained in (1), (3), (4), and (6) above.

Hence we find

$$(1)\ a = c \cdot \sin A, \qquad (2)\ a = b \cdot \tan A,$$
$$(3)\ a = c/\csc A, \qquad (4)\ a = b/\cot A.$$

If c and $\angle A$ are given we use formula (1) rather than (3), and if b and $\angle A$ are given we use formula (2) rather than (4).

23. Formulas giving side adjacent.—The side b is contained in (2), (3), (5), and (6) above.

Hence we find

$$(1)\ b = c \cdot \cos A, \qquad (2)\ b = a/\tan A,$$
$$(3)\ b = c/\sec A, \qquad (4)\ b = a \cdot \cot A.$$

If c and $\angle A$ are given we use (1) rather than (3), and if a and $\angle A$ are given we use formula (4) rather than (2).

24. Formulas giving the hypotenuse.—The side c is contained in (1), (2), (4), and (5) above.

Hence we find

$$(1)\ c = a/\sin A, \qquad (2)\ c = b/\cos A,$$
$$(3)\ c = a \cdot \csc A, \qquad (4)\ c = b \cdot \sec A.$$

25. Formulas in most frequent use.—The two formulas

$a = c \sin A$ (side opposite = hypotenuse x sine)
$b = c \cos A$ (side adjacent = hypotenuse x cosine)

are in frequent use in the further study of trigonometry and also in its applications to surveying and in mechanical problems.

26. Formulas for finding the functions of the angles.—To find the functions of the acute angles of a right triangle when its sides are given, the formulas required are those giving the definitions of the trigonometric functions. Thus, to find sin A in the right triangle ABC in which a, b, and c are given, use the formula $\sin A = a/c$, and to find tan A use the formula $\tan A = a/b$. In this manner all the functions of either acute angle of a right triangle may be found.

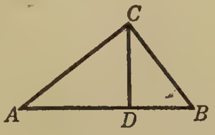
SIGHT WORK

1. State each of the following formulas in words:

(1) $a = c \sin A$, (2) $a = b \tan A$, (3) $b = c \cos A$
(4) $b = a \cot A$, (5) $c = a \csc A$, (6) $c = b \sec A$

(1) may be stated thus: *the side opposite either acute angle of a right triangle equals the hypotenuse times the sine of that angle.*

In the figure the triangles ABC, ADC, and BDC, are right angled. In each of the following give the most convenient formula for finding the part required.



Given	Required	Given	Required
2. $\angle A, AD$	DC	17. $\angle A, DC$	AD
3. $\angle B, BD$	DC	18. $\angle ACD, AC$	AD
4. $\angle ACD, AD$	DC	19. $\angle DCB, CB$	CD
5. $\angle DCB, CD$	DB	20. $\angle A, AC$	AD
6. $\angle A, AB$	AC	21. $\angle B, BC$	BD
7. $\angle B, AC$	AB	22. $\angle B, AB$	AC
8. $\angle DCB, DB$	CD	23. $\angle A, AD$	AC
9. $\angle A, AC$	CB	24. $\angle ACD, AD$	AC
10. $\angle B, BC$	AB	25. $\angle B, BD$	BC
11. $\angle ACD, CD$	AD	26. $\angle DCB, CD$	BC
12. $\angle B, DC$	BD	27. $\angle A, AB$	CB
13. $\angle A, CB$	AC	28. $\angle B, AC$	CB
14. AC, BC	tan A	29. DB, DC	tan B
15. AB, AC	cos A	30. DB, BC	cos B
16. AC, BC	cot A	31. BC, CD	cos $\angle DCB$

27. Units of measure for angles.—The principal unit of measure for angles is the **degree** which is one of the 90 equal parts of a right angle. One of the 60 equal parts of one degree is called a **minute** of angle, and one of the 60 equal parts of a minute is called a **second** of angle. Degrees, minutes, and seconds are written $^\circ$, $'$, $''$, respectively.

Thus 47 degrees 32 minutes 57 seconds is written $47^\circ\ 32'\ 57''$.

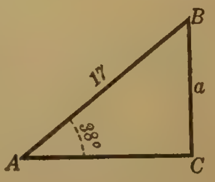
28. Use of tables in computing.—In solving triangles by means of the trigonometric functions, use is made of tables giving these functions for various angles. Such tables have been worked out for every minute of angle from 0° to 90° and in some cases for every second of angle.

Part of such a table is given here.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
30°	.5000	.8660	.5774	1.7321	1.1547	2.0000
31°	.5150	.8572	.6009	1.6643	1.1666	1.9416
32°	.5299	.8480	.6249	1.6003	1.1792	1.8871
33°	.5446	.8387	.6494	1.5399	1.1924	1.8361
34°	.5592	.8290	.6745	1.4826	1.2062	1.7883
35°	.5736	.8192	.7002	1.4281	1.2208	1.7434
36°	.5878	.8090	.7265	1.3764	1.2361	1.7013
37°	.6018	.7986	.7536	1.3270	1.2521	1.6616
38°	.6157	.7880	.7813	1.2799	1.2690	1.6243
39°	.6293	.7771	.8098	1.2349	1.2868	1.5890

The first column to the left in the table indicates the number of degrees in the angle. The next column gives the sines of these angles, the next gives the cosines, and so on.

The use of these tables may be illustrated by means of examples.



Example 1. Find sin 32°, cos 35°, tan 37°, cot 38°.

Solution: To find the sin 32° we look in the column headed Sin and in the line in which 32° stands and find the number .5299. Hence sin 32° = .5299. Similarly, we find cos 35° = .8192, tan 37° = .7536, and cot 38° = 1.2799.

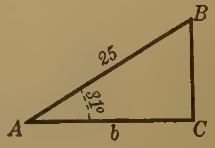
Example 2. If $\angle A = 38°$ and $c = 17$, find a.

Solution: Since $\sin A = a/c$, we obtain

$a = c \times \sin A = 17 \times \sin A$.

We now find in the table that sin 38° = .6157.

Hence $a = 17 \times .6157 = 10.4669$.



Example 3. If $\angle A = 31°$ and $c = 25$, find b.

Solution: Since $\cos A = b/c$ we have $b = c \times \cos A$, and hence $b = 25 \times .8572 = 21.43$.

Example 4. If $\angle A = 34°$ and $b = 12$, find c.

Solution: Since $\sec A = \frac{c}{b}$, we have $c = b \times \sec A = 12 \times 1.2062 = 14.474$.

29. Approximate character of tables and results.—With very few exceptions the numbers given in tables such as the one on page 22 are approximations. Sin $30^\circ = .5$ exactly and csc $30^\circ = 2$ exactly, but all the other numbers given in this table are approximations. Hence most of the results obtained by using such a table are only approximately correct; the closeness of the approximations depends upon the number of decimals in the table.

The following example shows another kind of approximation which is of importance in trigonometry:

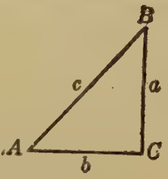
Example. Find the angle A to the nearest degree if $\sin A = .5774$.

Solution: Since $\sin 35^\circ = .5736$ and $\sin 36^\circ = .5878$, it follows that angle A lies between 35° and 36°. Further, since .5774 is nearer to .5736 than it is to .5878, we conclude that $\angle A$ is nearer 35° than 36°. Hence $\angle A$ to the nearest degree is 35°.

Later we shall learn to compute angles to the nearest minute and in some cases even to the nearest second.

EXERCISES

In the following exercises find the length of lines to the nearest one-hundredth. Each example refers to this figure.

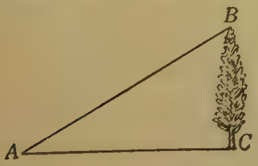


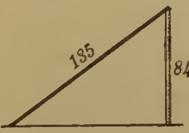
1. $\angle A = 35^\circ$, $c = 18$, find a.
2. $\angle A = 31^\circ$, $c = 24$, find b.
3. $\angle A = 37^\circ$, $a = 65$, find b.
4. $\angle A = 38^\circ$, $a = 42$, find c.
5. $\angle A = 32^\circ$, $b = 16$, find a.
6. $\angle A = 39^\circ$, $b = 38$, find c.
7. $\angle A = 33^\circ$, $c = 3.4$, find a.
8. $\angle A = 34^\circ$, $a = 1.34$, find c.

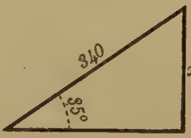
In the following six examples find the required angle to the nearest degree.

9. $a = 13$, $b = 17$, find $\angle A$.
10. $a = 34$, $b = 27$, find $\angle B$.
11. $a = 36$, $c = 60$, find $\angle A$.
12. $a = 39$, $c = 48$, find $\angle B$.
13. $b = 65$, $c = 78$, find $\angle A$.
14. $b = 5.7$, $c = 9.4$, find $\angle B$.

15. To find the height of a tree CB the line CA is measured, and also the angle CAB. Find the height if $\angle ACB = 90^\circ$, $CA = 124$ feet, and $\angle A = 31^\circ$.







16. From a point 84 feet above the ground a steel cable is fastened to a smokestack. What angle does this rope make with the ground if it is 135 feet long?

17. A boy's kite string is 340 feet long and makes an angle of 35° with the ground. Find the vertical height of the kite.

30. Finding functions of angles up to 45°.—The table on the page opposite gives four trigonometric functions for every degree of angle between 0° and 90°. The larger tables which will finally be used in practice give the functions for every minute of angle. However, the essential points to be mastered are the same in those tables as in the briefer one given here.

For all the angles up to 45° the number of degrees is indicated in the first column to the left. The functions sine, cosine, tangent, cotangent occur in columns in order from left to right. For example, to find cot 37° we look down the left column to 37° and then across the page to the column headed Cot at the top. That is, cot 37° = 1.3270.

31. Finding functions of angles between 45° and 90°.—For angles between 45° and 90° the number of degrees is given in the column to the right of the page. The column containing sines of angles less than 45° contains cosines of angles greater than 45°. Thus sin 40° is the cosine of 50°. The columns of tangents and cotangents are similarly related. This follows from the general principle that *cofunctions of complementary angles are equal.*

Thus, tan 65° = cot 25°, sin 72° = cos 18°, etc.

Sight Work on the Table

From the table on page 25 find:

1. sin 13°.
2. cos 42°.
3. sin 63°.
4. tan 42°.
5. sin 17°.
6. cos 29°.
7. cot 81°.
8. tan 67°.
9. cos 79°.
10. sin 84°.
11. cos 57°.
12. tan 54°.
13. cot 39°.
14. cos 63°.
15. tan 45°.
16. cot 17°.
17. sin 38°.
18. sin 68°.
19. tan 18°.
20. cot 10°.

Find the angle A to the nearest degree for each of the following:

21. sin A = .4846.
22. cos A = .3942.
23. tan A = 1.2782.
24. cot A = 1.6710.
25. tan A = 1.4920.
26. cot A = 1.9820.
27. tan A = .3902.
28. cot A = .9372.
29. sin A = .7381.
30. cos A = .7381.
31. cot A = 2.6420.
32. tan A = 1.9364.
33. cot A = 2.5628.
34. tan A = 2.14202.
35. sin A = .1948.

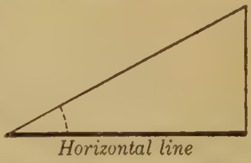
Using the table, compare the following and explain:

36. sin 63°, cos 27°.
37. cos 35°, sin 55°.
38. tan 41°, cot 49°.
39. cot 33°, tan 57°.
40. cot 13°, tan 77°.
41. tan 34°, cot 56°.

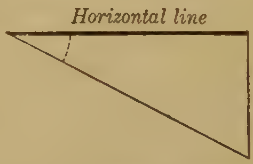
TABLE OF SINES, COSINES, AND TANGENTS

Angle	Sin	Cos	Tan	Cot	Angle
1°	.0175	.9998	.0175	57.2900	89°
2°	.0349	.9994	.0349	28.6363	88°
3°	.0523	.9986	.0524	19.0811	87°
4°	.0698	.9976	.0699	14.3007	86°
5°	.0872	.9962	.0875	11.4301	85°
6°	.1045	.9945	.1051	9.5144	84°
7°	.1219	.9925	.1228	8.1443	83°
8°	.1392	.9903	.1405	7.1154	82°
9°	.1564	.9877	.1584	6.3138	81°
10°	.1736	.9848	.1763	5.6713	80°
11°	.1908	.9816	.1944	5.1446	79°
12°	.2079	.9781	.2126	4.7046	78°
13°	.2250	.9744	.2309	4.3315	77°
14°	.2419	.9703	.2493	4.0108	76°
15°	.2588	.9659	.2679	3.7321	75°
16°	.2756	.9613	.2867	3.4874	74°
17°	.2924	.9563	.3057	3.2709	73°
18°	.3090	.9511	.3249	3.0777	72°
19°	.3256	.9455	.3443	2.9042	71°
20°	.3420	.9397	.3640	2.7475	70°
21°	.3584	.9336	.3839	2.6051	69°
22°	.3746	.9272	.4040	2.4751	68°
23°	.3907	.9205	.4245	2.3559	67°
24°	.4067	.9135	.4452	2.2460	66°
25°	.4226	.9063	.4663	2.1445	65°
26°	.4384	.8988	.4877	2.0503	64°
27°	.4540	.8910	.5095	1.9626	63°
28°	.4695	.8829	.5317	1.8807	62°
29°	.4848	.8746	.5543	1.8040	61°
30°	.5000	.8660	.5774	1.7321	60°
31°	.5150	.8572	.6009	1.6643	59°
32°	.5299	.8480	.6249	1.6003	58°
33°	.5446	.8387	.6494	1.5399	57°
34°	.5592	.8290	.6745	1.4826	56°
35°	.5736	.8192	.7002	1.4281	55°
36°	.5878	.8090	.7265	1.3764	54°
37°	.6018	.7986	.7536	1.3270	53°
38°	.6157	.7880	.7813	1.2799	52°
39°	.6293	.7771	.8098	1.2349	51°
40°	.6428	.7660	.8391	1.1918	50°
41°	.6561	.7547	.8693	1.1504	49°
42°	.6691	.7431	.9004	1.1106	48°
43°	.6820	.7314	.9325	1.0724	47°
44°	.6947	.7193	.9657	1.0355	46°
45°	.7071	.7071	1.0000	1.0000	45°
Angle	Cos	Sin	Cot	Tan	Angle

32. Angles of elevation and of depression.—In many problems in practical measurements the angle between a horizontal line and the line from the observer to a certain elevated object is measured. Such angles are called *angles of elevation.*



In a similar way, when the observer is above the object at which he is looking the angle between the horizontal line and a line sloping downward to the object is measured. This angle is called the *angle of depression.*



Angles of elevation and angles of depression occur frequently in practical surveying. Thus, the height of a tree, of a building, or of a mountain may be obtained by finding its angle of elevation from a given point, when certain additional data are also known or may be found without ascending the object whose height is to be measured.

Similarly, the depth of a canyon may be measured by an observer standing above it at a point from which the bottom of the canyon may be seen by simply measuring the angle of depression from the point where he is standing and obtaining certain other data which can be found without descending the canyon. For this problem see Example 7, page 165.

33. Solving a triangle.—Solving a triangle consists in finding the numerical measures of the parts not given.

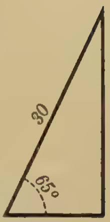
EXERCISES

In each of the following solve the right triangle ABC with sides a, b, c if $\angle C$ is the right angle. Find each angle to the nearest degree.

1. $a = 7$, $b = 12$.
2. $a = 28$, $b = 17$.
3. $a = 60$, $c = 75$.
4. $a = 14$, $c = 50$.
5. $a = 27$, $\angle A = 46°$.
6. $a = 3.72$, $\angle A = 68°$.
7. $a = .94$, $\angle B = 14°$.
8. $a = 7.94$, $\angle B = 74°$.
9. $b = 97$, $c = 180.6$.
10. $b = 3.9$, $c = 7.8$.
11. $b = 8.7$, $\angle A = 53°$.
12. $b = 34.8$, $\angle A = 81°$.
13. $b = 4.35$, $\angle B = 17°$.
14. $b = 264$, $\angle B = 59°$.
15. $c = 94.1$, $\angle A = 42°$.
16. $c = 7.64$, $\angle A = 71°$.
17. $c = 8.3$, $\angle B = 57°$.
18. $c = 58.6$, $\angle B = 38.°$
19. $\angle A = 47°$, $b = 17.4$.
20. $\angle B = 34°$, $c = 62.8$.
21. $a = 39.1$, $c = 87.2$.
22. $b = 5.42$, $c = 17.4$.
23. $a = 8.92$, $\angle B = 43°$.
24. $\angle B = 29°$, $b = 54.7$.

PROBLEMS: GROUP A

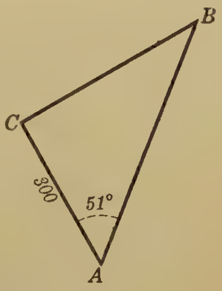
1. A ladder 30 feet long rests against a building and makes an angle of 65° with the ground. Find the height above the ground at which it touches the building.



2. A rope 48 feet long reaches from the top of a flag pole to a point on the ground 18 feet from its base. Find to the nearest degree the angle which it makes with the ground. How high is the flag pole?

3. The shadow cast on a horizontal plane by a vertical pole 16 feet high is 25.5 feet long. Find to the nearest degree the angle of elevation of the sun.

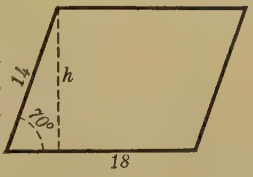
4. Two points, C and B, are on opposite sides of a stream. A line CA at right angles to CB is measured 300 feet long, and the angle CAB is found by measurement to be 51°. What is the distance CB?



5. A guy rope 37 feet long is attached to a telephone pole at a height of 26 feet from the ground. Determine to the nearest degree the angle which it makes with the pole.

6. Two sides of a parallelogram are 14 and 18 feet, respectively, and their included angle is 70°. Find the area of the parallelogram.

Suggestion: In the figure, find h, the altitude of the parallelogram; then use the usual formula giving the area.

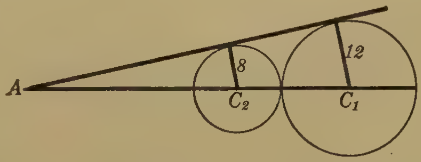


7. The shadow of a tree is observed to be 60 feet long, and at the same time the shadow of a seven-foot pole is found to be $7\frac{2}{3}$ feet long. Find to the nearest degree the angle of elevation of the sun, and determine the height of the tree.

8. A building seen at a distance of 350 feet has an angle of elevation of 41°. Find the height of the building.

9. The angle of depression of a point as seen from the top of a building 75 feet high is 15°. How far distant from the base of the building is this point?

10. Two circles of radii 8 inches and 12 inches are tangent externally. What angle does a common external tangent make with the line of centers?



Suggestion: Let C_1 and C_2 be the centers of the circles. Let the line of centers meet the common tangent at A. The distance between the centers is 20 inches. Hence

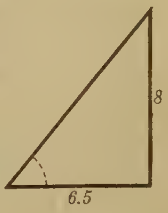
$$\frac{AC_2}{AC_2 + 20} = \frac{8}{12}.$$

PROBLEMS: GROUP B

1. A ladder 24 feet long is placed against a building so as to touch a point 18 feet above the ground. How far from the building is the foot of the ladder if the building stands on horizontal ground?

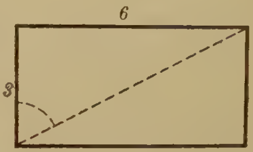
2. The shadow cast on the earth by a vertical pole 8 feet long is 6.5 feet. Find to the nearest degree the altitude of the sun.

3. A rope 60 feet long reaches from the top of a flag pole to a point on the level ground 40 feet from the pole. What angle does this rope make with the ground? How tall is the flag pole?

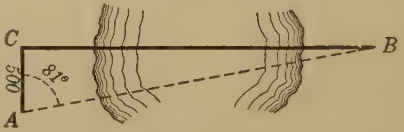


4. A rectangle is 3 feet wide and 6 feet long. Determine to the nearest degree the angle between the diagonal and a short side of this rectangle.

5. A road rises 15 feet in a horizontal distance of 250 feet. Find the angle of elevation of this road to the nearest degree.

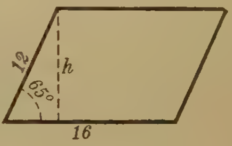


6. Two points, C and B, are on opposite sides of a lake. A line CA, 500 feet long, is measured along a straight road which runs at right angles to the line CB. The angle at A (angle CAB) is found to be 81°. Find the distance from C to B.

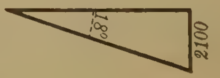


7. Two sides of a parallelogram are 12 and 16 feet, respectively, and the angle included between them is 65°. Find the area of the parallelogram.

Suggestion: In the figure find h, the altitude of the parallelogram, and then find its area in the usual manner.



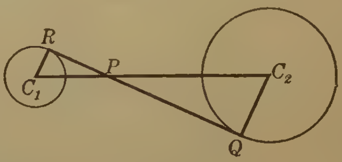
8. The angle of depression of a point in a plane as seen from a mountain is 18°. What is the air-line distance from this point to the point of observation if it is known that the top of the mountain is 2100 feet above the plane?



9. The length of a kite string is 500 feet. Find the height of the kite if its angle of elevation from the point where the string is held is 47°.

10. Two circles of radii 4 inches and 10 inches, respectively, have their centers 28 inches apart. What angle does an internal common tangent make with the line of centers?

Suggestion: Notice that triangles PRC_1 and PQC_2 are similar and that hence $C_1R : C_2Q = C_1P : PC_2$. From this relation and $C_1P + PC_2 = 28$ find that $C_1P = 8$ and $PC_2 = 20$.



11. Two circles with radii r_1 and r_2 inches have their centers a inches apart. Derive a formula giving the sine of the angle which their line of centers makes with an internal common tangent.

$r_1 + r_2$

CHAPTER IV

LOGARITHMS

IN APPLYING the formulas of trigonometry to the solution of problems long computations occur frequently. The laboriousness of these computations is greatly reduced by the use of logarithms, which enable us to replace multiplication and division respectively by addition and subtraction.

34. Definition of logarithms.—The index of the power to which a given number called the **base** must be raised to equal a second number is called the **logarithm** of the second number.

That is, x is the logarithm of N to the base a if

$$a^x = N \text{ (exponential notation)}$$

This is also denoted by writing

$$x = \log_a N \text{ (logarithmic notation)}$$

Thus, since $2^4 = 16$, $4 = \log_2 16$; since $10^2 = 100$, $2 = \log_{10} 100$; since $10^3 = 1{,}000$, $3 = \log_{10} 1000$. In general if $y^z = k$, then $z = \log_y k$.

35. The base.—In developing a table of logarithms the same base must be used for all numbers. While any positive number except 1 might be used as a base, in practice the number 10 is the only one used for tables designed as aids in computation.

36. Systems of logarithms.—The system of logarithms having 10 as its base is called the **Common System of Logarithms.** Another system, called the *Napierian or Natural* System, is used for theoretical purposes. Its base cannot be expressed exactly in decimal notation. To seven decimal places it is 2.7182818. This base is usually denoted by e.[1]

In this book the common system only is used. Since the base of all logarithms in that system is 10, the base will be omitted in our notation. Thus $\log N$ is understood to mean $\log_{10} N$. If $x = \log N$, it is understood that $10^x = N$.

---

[1] The use of 10 as the base of logarithms for practical computation is natural, since it fits more smoothly into our decimal number system than any other base. The reason for using e as a base for theoretical purposes cannot be explained here. It becomes apparent when we come to study the calculus.

37. Laws of exponents.—Since logarithms are exponents, they have the properties (that is, they obey the laws) of exponents. The following laws of exponents are needed in developing the rules for logarithmic computation:

$$
\begin{array}{llll}
(1) & a^x \cdot a^y = a^{x+y} & (4) & \sqrt[y]{a^x} = a^{x/y} \\
(2) & a^x \div a^y = a^{x-y} & (5) & a^{-x} = \dfrac{1}{a^x} \\
(3) & (a^x)^y = a^{xy} & (6) & a^\circ = 1
\end{array}
$$

Thus,

$$
\begin{array}{llll}
(1') & 2^2 \cdot 2^3 = 2^{2+3} = 2^5 & (4') & \sqrt[3]{2^6} = 2^{\frac{6}{3}} = 2^2 \\
(2') & 2^5 \div 2^2 = 2^{5-2} = 2^3 & (5') & 2^{-3} = \dfrac{1}{2^3} \\
(3') & (2^2)^3 = 2^{2\cdot 3} = 2^6 & (6') & 2^\circ = 1
\end{array}
$$

38. Logarithm of a product.—The logarithm of a product is the sum of the logarithms of the factors.

Take 10 as the base and assume that

$$M = 10^x,\ N = 10^y.$$

Then, $\log M = x,\ \log N = y.$

By the laws of exponents $M \cdot N = 10^x \cdot 10^y = 10^{x+y}.$

Hence $x + y = \log MN.$

That is, $\log MN = \log M + \log N.$

Similarly, if $P = 10^z$,

$$M \cdot N \cdot P = 10^x \cdot 10^y \cdot 10^z = 10^{x+y+z}$$

and $\log (M \cdot N \cdot P) = \log M + \log N + \log P.$

This law evidently holds for any base and for any number of factors. Hence we have the rule:

To find the logarithm of a product, find the sum of the logarithms of the factors.

39. Logarithm of a quotient.—The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.

If M and N are the same as in the preceding paragraph, we have

$$\frac{M}{N} = \frac{10^x}{10^y} = 10^{x-y}.$$

That is, $\log \dfrac{M}{N} = x - y = \log M - \log N.$

Hence, we have the rule:

To find the logarithm of a quotient, subtract the logarithm of the divisor from the logarithm of the dividend.

40. Logarithm of a power.—The logarithm of a power of a number equals the logarithm of the number multiplied by the index of the power.

If $M = 10^x$, then $M^y = (10^x)^y = 10^{xy}$. Hence, the logarithm of M^y is xy, or $\log M^y = y \log M$, and we have the rule:

To find the logarithm of a power of a number multiply the logarithm of the number by the index of the power.

41. Logarithm of a root.—The rule of the preceding paragraph holds also when the index of the power is a fraction.

By (4) of §37, $\sqrt[q]{M} = M^{\frac{1}{q}}$. Hence by the rule of §40, $\log \sqrt[q]{M} = \log M^{\frac{1}{q}} = \frac{1}{q} \log M$, and we have the rule:

To find the logarithm of a root of a number, divide the logarithm of the number by the index of the root.

42. Negative and zero logarithms.—From (5) and (6) of §37, two important results regarding logarithms are obtained.

If the base a of the system of logarithms used is greater than 1, as in the common system, then a^x, where x is positive (whether integral or fractional), is always greater than 1, and $1/a^x$ is accordingly less than 1. Consequently if

$$M = \frac{1}{a^x} = a^{-x}, \text{ then, } \log M = -x.$$

That is,

The logarithm of any positive number less than 1 *is negative.*

From (6) of §37, $10^0 = 1$, and hence, $\log 1 = 0$.

SIGHT WORK

1. Given, $\log 43 = 1.63347$ and $\log 76 = 1.88081$, find $\log(43 \times 76)$ and also $\log 76 \div 43$.

2. Given, $\log 21 = 1.32222$, $\log 186 = 2.26951$, and $\log 324 = 2.51055$, find $\log(21 \times 186 \times 324.)$

3. From the logarithms given in Example 2, find $\log \frac{324 \times 186}{21}$.

Suggestion: First find $\log 324 \times 186$.

4. From the logarithms in Example 2, find $\log \sqrt[3]{186 \times 21}$.

5. Given, $\log 1.06 = 0.02531$, find $\log (1.06)^{16}$.

43. Approximate logarithms.—The logarithms of all rational numbers other than 1, 10, 100 . . . and their reciprocals, can be given only approximately by means of decimals or common fractions. The closeness of the approximation depends upon the number of decimal places to which a table is carried. (See page 38.)

Different tables are used for different purposes. Thus there are tables with four, five, six, seven, and even ten or more places.

44. Characteristic and mantissa.—The integral part of a logarithm is called the **characteristic** and the decimal part is called the **mantissa.**

Thus, in the logarithm 3.78104, 3 is the characteristic, and .78104 is the mantissa.

45. Rule for characteristics.—From the following table many important inferences may be drawn. Note that the equations in the second column follow immediately from those in the first.

$$\begin{array}{lrlr}
10^4 = & 10{,}000 & \log 10{,}000 & = 4 \\
10^3 = & 1000 & \log 1000 & = 3 \\
10^2 = & 100 & \log 100 & = 2 \\
10^1 = & 10 & \log 10 & = 1 \\
10^0 = & 1 & \log 1 & = 0 \\
10^{-1} = & .1 & \log .1 & = -1 \\
10^{-2} = & .01 & \log .01 & = -2 \\
10^{-3} = & .001 & \log .001 & = -3 \\
10^{-4} = & .0001 & \log .0001 & = -4
\end{array}$$

From this table it is evident that if a number lies between 1000 and 10,000, its logarithm lies between 3 and 4, and may be written as $3 +$ a fraction. Such a number has 4 digits preceding the decimal point. Similarly, a number between 100 and 1000 has 3 digits and its logarithm is $2 +$ a fraction. Again, the logarithm of a number of 1 digit is $0 +$ a fraction. Hence we have:

RULE I. *If a number is greater than* 1, *the characteristic of its logarithm is positive and is one less than the number of digits preceding the decimal point.*[1]

Again, from the table it is evident that if a number lies between 1 and .1, its logarithm lies between -1 and 0; it may thus be written as $-1 +$ a fraction. For a number lying between .1 and .01, the logarithm is between -2 and -1, and may be written as $-2+$ a fraction. Now any number between .1 and .01, as .0831 or .01001, has one zero immediately following the decimal point, while a number between 1 and .1 has no zero immediately following the decimal point. Continuing in this way, it appears that the characteristic of the logarithm of a number having *two* zeros immediately following the decimal point is -3; for one having three such zeros, it is -4; and so on. Hence, we have:

---

[1] Zero is here regarded as a positive number.

RULE II. *If a positive number is less than* 1, *the characteristic of its logarithm is negative, and is numerically one greater than the number of zeros immediately following the decimal point.*

46. Ways of writing negative characteristics.—From any table of logarithms it can be found that the mantissa of the logarithm of 347 is .54033. From Rule I above, the characteristic is 2. Hence,

$$\log 347 = 2.54033.$$

If, however, we desire to write the logarithm of .0347, we must recognize that that logarithm is made up of the characteristic -2 (by Rule II above) plus a positive fraction (in this case .54033 again). It would plainly be misleading to write this as -2.54033. It may be written as $.54033 - 2$, but is more often written $\bar{2}.54033$, the minus sign being written above the 2 to indicate that only the 2, and not the decimal, is affected.

In computation a negative characteristic is often replaced by some integer minus 10.

Thus, since

$$\begin{aligned} -2 &= 8 - 10, \\ \log .0347 &= .54033 - 2 \\ &= .54033 + 8 - 10 \\ &= 8.54033 - 10 \end{aligned}$$

Likewise, -1 is replaced by $9 - 10$, as in

$$.27803 - 1 = \bar{1}.27803 = 9.27803 - 10,$$

and similarly in other cases.

47. The significant part of a number.—If all zeros at the beginning and the end of a number and its decimal point are omitted, the remaining part is called the **significant part** of the number.

Thus the significant part of 63,800 is 638, the significant part of .0102 is 102, the significant part of 0.014 is 14, the significant part of 40.08 is 4008, and the significant part of 3.41 is 341.

48. Numbers having the same mantissa.—From §46 we have

$$\log 347 = 2.54033.$$

Since, $3470 = 347 \times 10,$

it follows that
$$\begin{aligned} \log 3470 &= \log 347 + \log 10 \ (\S 38) \\ &= 2.54033 + 1 = 3.54033. \end{aligned}$$

Also, $34.7 = 347 \div 10$

and hence
$$\begin{aligned} \log 34.7 &= \log 347 - \log 10 \ (\S 39) \\ &= 2.54033 - 1 = 1.54033 \end{aligned}$$

In this manner the following table is easily constructed:

$$\begin{array}{llll} \log 3.47 = 0.54033 & & \log .347 = \bar{1}.54033 \\ \log 34.7 = 1.54033 & & \log .0347 = \bar{2}.54033 \\ \log 347 = 2.54033 & & \log .00347 = \bar{3}.54033 \\ \log 3470 = 3.54033 & & \log .000347 = \bar{4}.54033 \end{array}$$

Hence we conclude that *the logarithms of numbers having the same significant parts have the same mantissa.*

SIGHT WORK

Given, log 1784 = 3.25139, find the logarithms of 178.4, 17.84, 1.784, 0.1784, .01784, 17840.

49. Logarithms which are read directly from table.—In the table on the page opposite, the first column to the left is the *number column.* The other columns contain logarithms.

To avoid crowding the page, repetition of numbers is avoided in printing. Thus in the first column of logarithms, 48 is omitted from all numbers between 48001 and 49136. These figures must be supplied in using the table.

Example 1. Find log 33.7.

Solution: In the line of 337 and in the first column we find 763. Looking at the next full five-place number above in this column, we find that the complete mantissa is 52763. Hence, by §45 log 33.7 = 1.52763.

Example 2. Find log 327.8.

Solution: In the line of 327 and in the column which has the figure 8 at the top we find 561, before which 51 must be supplied. Hence 51561 is the required mantissa, and log 327.8 = 2.51561.

In the first column of logarithms are the logarithms of numbers with three significant figures. The other columns contain logarithms of numbers with four significant figures, the particular column depending upon the fourth figure of the number whose logarithm is sought. Thus, log 3172 is in the column headed 2, log 3217 is in the column headed 7, and so on.

Logarithms 300—350

Numbers 300–350 Logs 47712–54518

N	0	1	2	3	4	5	6	7	8	9	P.P.
300	47 712	47 727	47 741	47 756	47 770	47 784	47 799	47 813	47 828	47 842	15
301	857	871	885	900	914	929	943	958	972	986	1 1.5
302	48 001	48 015	48 029	48 044	48 058	48 073	48 087	48 101	48 116	48 130	2 3.0
303	144	159	173	187	202	216	230	244	259	273	3 4.5
304	287	302	316	330	344	359	373	387	401	416	4 6.0
305	430	444	458	473	487	501	515	530	544	558	5 7.5
306	572	586	601	615	629	643	657	671	686	700	6 9.0
307	714	728	742	756	770	785	799	813	827	841	7 10.5
308	855	869	883	897	911	926	940	954	968	982	8 12.0
309	996	49 010	49 024	49 038	49 052	49 066	49 080	49 094	49 108	49 122	9 13.5
310	49 136	49 150	49 164	49 178	49 192	49 206	49 220	49 234	49 248	49 262	14
311	276	290	304	318	332	346	360	374	388	402	1 1.4
312	415	429	443	457	471	485	499	513	527	541	2 2.8
313	554	568	582	596	610	624	638	651	665	679	3 4.2
314	693	707	721	734	748	762	776	790	803	817	4 5.6
315	831	845	859	872	886	900	914	927	941	955	5 7.0
316	969	982	996	50 010	50 024	50 037	50 051	50 065	50 079	50 092	6 8.4
317	50 106	50 120	50 133	147	161	174	188	202	215	229	7 9.8
318	243	256	270	284	297	311	325	338	352	365	8 11.2
319	379	393	406	420	433	447	461	474	488	501	9 12.6
320	50 515	50 529	50 542	50 556	50 569	50 583	50 596	50 610	50 623	50 637	
321	651	664	678	691	705	718	732	745	759	772	
322	786	799	813	826	840	853	866	880	893	907	
323	920	934	947	961	974	987	51 001	51 014	51 028	51 041	
324	51 055	51 068	51 081	51 095	51 108	51 121	135	148	162	175	
325	188	202	215	228	242	255	268	282	295	308	
326	322	335	348	362	375	388	402	415	428	441	
327	455	468	481	495	508	521	534	548	561	574	
328	587	601	614	627	640	654	667	680	693	706	
329	720	733	746	759	772	786	799	812	825	838	
330	51 851	51 865	51 878	51 891	51 904	51 917	51 930	51 943	51 957	51 970	13
331	983	996	52 009	52 022	52 035	52 048	52 061	52 075	52 088	52 101	1 1.3
332	52 114	52 127	140	153	166	179	192	205	218	231	2 2.6
333	244	257	270	284	297	310	323	336	349	362	3 3.9
334	375	388	401	414	427	440	453	466	479	492	4 5.2
335	504	517	530	543	556	569	582	595	608	621	5 6.5
336	634	647	660	673	686	699	711	724	737	750	6 7.8
337	763	776	789	802	815	827	840	853	866	879	7 9.1
338	892	905	917	930	943	956	969	982	994	53 007	8 10.4
339	53 020	53 033	53 046	53 058	53 071	53 084	53 097	53 110	53 122	135	9 11.7
340	53 148	53 161	53 173	53 186	53 199	53 212	53 224	53 237	53 250	53 263	12
341	275	288	301	314	326	339	352	364	377	390	1 1.2
342	403	415	428	441	453	466	479	491	504	517	2 2.4
343	529	542	555	567	580	593	605	618	631	643	3 3.6
344	656	668	681	694	706	719	732	744	757	769	4 4.8
345	782	794	807	820	832	845	857	870	882	895	5 6.0
346	908	920	933	945	958	970	983	995	54 008	54 020	6 7.2
347	54 033	54 045	54 058	54 070	54 083	54 095	54 108	54 120	133	145	7 8.4
348	158	170	183	195	208	220	233	245	258	270	8 9.6
349	283	295	307	320	332	345	357	370	382	394	9 10.8
350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	

50. Antilogarithms.—A number is called the **antilogarithm** (written antilog) of its logarithm.

Thus, if $a = \log N$, then $N = \text{antilog } a$.

51. Finding Antilogarithms.—If the given mantissa occurs exactly in the table, the corresponding number (the antilog) can be found at once.

Thus, if the given mantissa is 68538 we find at once that the significant part of the required number is 4846.

When the mantissa does not occur exactly in the table the process of finding the number is shown in the following example:

Example. Find antilog 1.66413.

Solution: We first find the significant part of the number. For this purpose the characteristic is disregarded. From the first column of logarithms we see that the given mantissa lies between the mantissas of 461 and 462. Hence, we write 461 as the first part of the required number. Passing along the line of 461, we see that the given mantissa lies between the mantissas of 4614 and 4615. Hence we annex the number 4 to 461, obtaining 4614.

The tabular difference at this point (417—408) is 9 and the given mantissa is 5 greater than the mantissa of 4614. Looking in the table of proportional parts under 9, we find that a tabular difference of 5 corresponds most nearly to the extra digit 6. Hence we annex 6 to 4614, obtaining 46,146 as the significant part of the required number.

Hence, by §45, antilog 1.66413 = 46.146.

NOTE.—In the above example the given mantissa, 66413, lies 5/9 or .556 of the way from 66408 to 66417 (the mantissa of 4614 and 4615, respectively). Hence, if conditions warranted and required that degree of accuracy, 556 should be annexed to 4614. However, a five-place table admits of error even in the fifth place. Moreover, the data in any ordinary practical problem will contain errors which make it useless to attempt a higher degree of accuracy in computation. See pages 142–144.

EXERCISES

Find the antilogs of the following:

1. 1.65427.
2. 2.66671.
3. 3.67779.
4. 0.68314.
5. $\bar{1}.69179$.
6. 1.65394.
7. 2.67194.
8. $\bar{1}.68034$.
9. $\bar{2}.69137$.
10. 0.66491.
11. 1.66902.
12. 0.67207.
13. 0.69100.
14. $\bar{2}.68002$.
15. $\bar{3}.65490$.
16. 0.69012.
17. $\bar{4}.67204$.
18. $\bar{2}.68300$.
19. $\bar{2}.67094$.
20. $\bar{1}.66714$.
21. 2.68013.
22. $\bar{1}.69004$.
23. $\bar{2}.65940$.
24. $\bar{3}.67894$.

Logarithms 450—500

Numbers 450–500 Logs 65321–69975

N	0	1	2	3	4	5	6	7	8	9	P.P.
450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408	10
451	418	427	437	447	456	466	475	485	495	504	1 1.0
452	514	523	533	543	552	562	571	581	591	600	2 2.0
453	610	619	629	639	648	658	667	677	686	696	3 3.0
454	706	715	725	734	744	753	763	772	782	792	4 4.0
455	801	811	820	830	839	849	858	868	877	887	5 5.0
456	896	906	916	925	935	944	954	963	973	982	6 6.0
457	992	66 001	66 011	66 020	66 030	66 039	66 049	66 058	66 068	66 077	7 7.0
458	66 087	096	106	115	124	134	143	153	162	172	8 8.0
459	181	191	200	210	219	229	238	247	257	266	9 9.0
460	66 276	66 285	66 295	66 304	66 314	66 323	66 332	66 342	66 351	66 361	
461	370	380	389	398	408	417	427	436	445	455	
462	464	474	483	492	502	511	521	530	539	549	
463	558	567	577	586	596	605	614	624	633	642	
464	652	661	671	680	689	699	708	717	727	736	
465	745	755	764	773	783	792	801	811	820	829	
466	839	848	857	867	876	885	894	904	913	922	
467	932	941	950	960	969	978	987	997	67 006	67 015	
468	67 025	67 034	67 043	67 052	67 062	67 071	67 080	67 089	099	108	
469	117	127	136	145	154	164	173	182	191	201	
470	67 210	67 219	67 228	67 237	67 247	67 256	67 265	67 274	67 284	67 293	9
471	302	311	321	330	339	348	357	367	376	385	1 0.9
472	394	403	413	422	431	440	449	459	468	477	2 1.8
473	486	495	504	514	523	532	541	550	560	569	3 2.7
474	578	587	596	605	614	624	633	642	651	660	4 3.6
475	669	679	688	697	706	715	724	733	742	752	5 4.5
476	761	770	779	788	797	806	815	825	834	843	6 5.4
477	852	861	870	879	888	897	906	916	925	934	7 6.3
478	943	952	961	970	979	988	997	68 006	68 015	68 024	8 7.2
479	68 034	68 043	68 052	68 061	68 070	68 079	68 088	097	106	115	9 8.1
480	68 124	68 133	68 142	68 151	68 160	68 169	68 178	68 187	68 196	68 205	
481	215	224	233	242	251	260	269	278	287	296	
482	305	314	323	332	341	350	359	368	377	386	
483	395	404	413	422	431	440	449	458	467	476	
484	485	494	502	511	520	529	538	547	556	565	
485	574	583	592	601	610	619	628	637	646	655	
486	664	673	681	690	699	708	717	726	735	744	
487	753	762	771	780	789	797	806	815	824	833	
488	842	851	860	869	878	886	895	904	913	922	
489	931	940	949	958	966	975	984	993	69 002	69 011	
490	69 020	69 028	69 037	69 046	69 055	69 064	69 073	69 082	69 090	69 099	8
491	108	117	126	135	144	152	161	170	179	188	1 0.8
492	197	205	214	223	232	241	249	258	267	276	2 1.6
493	285	294	302	311	320	329	338	346	355	364	3 2.4
494	373	381	390	399	408	417	425	434	443	452	4 3.2
495	461	469	478	487	496	504	513	522	531	539	5 4.0
496	548	557	566	574	583	592	601	609	618	627	6 4.8
497	636	644	653	662	671	679	688	697	705	714	7 5.6
498	723	732	740	749	758	767	775	784	793	801	8 6.4
499	810	819	827	836	845	854	862	871	880	888	9 7.2
500	69 897	69 906	69 914	69 923	69 932	69 940	69 949	69 958	69 966	69 975	
N	0	1	2	3	4	5	6	7	8	9	

52. Interpolation.—The process used on the preceding page for finding a number when the given mantissa does not occur in the table is called **interpolation.** In finding numbers whose logarithms are given, this process must be used very frequently, since only in exceptional cases does a given mantissa occur in the table.

The process of interpolation must also be used in finding the logarithms of given numbers when these contain more than four significant figures, since in this case the logarithms cannot be read directly from the table.

Example. Find log 32,714.

Solution: Using the table on page 35 we find

$$\log 32710 = 4.51468$$

and

$$\log 32720 = 4.51481.$$

The number 32,714 lies four-tenths of the way from 32,710 to 32,720. A reasonable approximation to its logarithm is the number four-tenths of the way from 4.51468 to 4.51481. The difference of these logarithms is 13 (hundred-thousandths); four-tenths of 13 is 5.2; hence we add 5 to the mantissa 51468 and have log 32714 = 4.51473.

In the column at the right of the table we find that for a tabular difference of 13, an extra digit of 4 corresponds to 5.2. Use of the tabular difference lists (called table of proportional parts) often shortens this work.

In working with a five-place table do not extend the figures in the mantissa beyond that number.

SIGHT WORK

Find the logarithms of the following:

1. 325.47
2. 47323
3. 3.2001
4. 309.84
5. 4985.3
6. 336.38
7. 4897.4
8. 3016.3
9. 31.715
10. 4895.4
11. 32247
12. 4.5956
13. 4.6324
14. 30.059
15. 456.91
16. 34.752

53. Errors in interpolation.—Using a six-place table, we find that the logarithm of 32,714 is 4.514734. Hence whatever error may be due to interpolation, it does not in this case affect the mantissa in the last place when working with a five-place table. With unimportant exceptions, it is true that when interpolating between any two consecutive mantissas given in a table, the result will be correct to the number of places of that table.

However, if we attempt to find the logarithm of a number by interpolation between two mantissas that are far apart there will be serious error. Thus, suppose we interpolate between log 200 = 2.30103 and log 300 = 2.47712 to find log 250, we shall have log 250 = 2.38908, whereas in reality log 250 = 2.39794.

54. Operations on logarithms.—Any computation by means of logarithms consists essentially of three steps.

(1) Finding logarithms of numbers given in the problem.

(2) Combining these logarithms by means of addition, subtraction, multiplication, and division, thus obtaining the logarithm of the desired result.

(3) Finding the number corresponding to this final logarithm.

Steps (1) and (3) have already been studied in §§ 48–53. The only difficulty in step (2) arises from the fact that the mantissa is *always* positive, while the characteristic may be positive or negative, but must *always* be an integer.

Thus, it is not so clear how $\bar{2}.79426$ may be divided by 3 in such a way as to make the characteristic of the result an integer and the mantissa positive.

In the following paragraphs we shall study the operations on logarithms and the various devices used to adapt them to the requirements of practical computation.

55. Adding logarithms.—Adding logarithms when the characteristics are positive is exactly the same as adding ordinary decimals. When one or more characteristics are negative the process is shown by the following examples:

Example 1. Add 2.47162 and $\bar{2}.86473$.

Solution: Rewriting the second logarithm by adding and subtracting 10, we have 8.86473 − 10.

The addition is now performed as in algebra. The characteristic is 11 − 10, or + 1. The result may then be written 1.33635.

$$\begin{array}{r} 2.47162 \\ \underline{8.86473 - 10} \\ 11.33635 - 10 \\ \text{or } 1.33635 \end{array}$$

Example 2. Add 1.40198, 0.16432 and $\bar{3}.66219$.

Solution: After rewriting the third logarithm we have the example in addition shown at the right.

$$\begin{array}{r} 1.40198 \\ 0.16432 \\ \underline{7.66219 - 10} \\ 9.22849 - 10 \end{array}$$

The characteristic in the result is 9 − 10, or − 1, and the result may therefore be written $\bar{1}.22849$. However, this change is not necessary.

56. Subtracting logarithms.—The following examples show the various combinations that arise in subtracting logarithms.

Example 1. Subtract 2.34718 from 4.16782.

In this case the subtraction differs in no respect from the subtraction of ordinary decimals.

$$\begin{array}{r} 4.16782 \\ \underline{2.34718} \\ 1.82064 \end{array}$$

Example 2. Subtract 2.17142 from 1.29476.

Solution: In place of 1.29476 write 11.29476 − 10. We then have the operation shown at the right.

$$\begin{array}{r} 11.29476 - 10 \\ 2.17142 \phantom{{}-10} \\ \hline 9.12334 - 10 \end{array}$$

For use in finding the antilogarithm it is to be noted that the characteristic is − 1, but no change in the written form is required.

Example 3. Subtract $\bar{2}$.14762 from 1.98764.

Solution: Rewriting both in the form suggested above, we have

$$\begin{array}{r} 11.98764 - 10 \\ 8.14762 - 10 \\ \hline 3.84002 \phantom{{}-10} \end{array}$$

Example 4. Subtract $\bar{1}$.62387 from $\bar{2}$.27941.

Solution: Proceed exactly as in the preceding, except that in the minuend 18 − 20 instead of 8 − 10 should be used.

$$\begin{array}{r} 18.27941 - 20 \\ 9.62387 - 10 \\ \hline 8.65554 - 10 \end{array}$$

57. Multiplying logarithms.—This presents no difficulty when the characteristic is positive. If the characteristic is negative, it may be expressed as some number less 10, and then the multiplying is done as in algebra.

Example: Multiply $\bar{3}$.84361 by 3.

The solution is shown at the right.

$$\begin{array}{rr} & 7.84361 - 10 \\ & 3 \\ \hline & 23.53083 - 30 \\ = & 3.53083 - 10 \\ = & \bar{7}.53083 \phantom{{}-10} \end{array}$$

58. Dividing a logarithm.—Difficulty arises when the characteristic is negative. To the characteristic should be added and subtracted some number which is exactly divisible by the number by which the logarithm is to be divided.

Example: Divide $\bar{1}$.84762 by 3.

Solution (1): Add and subtract 30. We then have 29.84762 − 30 to be divided by 3. The quotient is 9.94921 − 10, or $\bar{1}$.94921.

Solution (2): Add and subtract 3. Then 2.84762 − 3 divided by 3 gives .94921 − 1 or $\bar{1}$.94921, as before.

SIGHT WORK

Add and also subtract:

1. $\begin{array}{r} 4.19178 \\ \underline{1.47214} \end{array}$

2. $\begin{array}{r} 3.86421 \\ \underline{\bar{1}.93261} \end{array}$

3. $\begin{array}{r} 2.62191 \\ \underline{3.45231} \end{array}$

4. $\begin{array}{r} 1.97162 \\ \underline{2.14645} \end{array}$

5. $\begin{array}{r} 1.61241 \\ \underline{\bar{3}.71592} \end{array}$

6. $\begin{array}{r} \bar{4}.64825 \\ \underline{\bar{1}.91476} \end{array}$

7. $\begin{array}{r} \bar{3}.45264 \\ \underline{2.89317} \end{array}$

8. $\begin{array}{r} 3.91724 \\ \underline{\bar{5}.61987} \end{array}$

Multiply and also divide:

9. 3.48265 by 3.
10. 2.78436 by 3.
11. $\bar{1}$.18136 by 4.
12. $\bar{4}$.82671 by 2.
13. 3.61938 by 3.
14. $\bar{3}$.19641 by 2.
15. $\bar{1}$.81437 by 3.
16. 2.85723 by 4.
17. $\bar{5}$.17143 by 2.

59. Computation by means of logarithms.—The examples on this page illustrate the uses of logarithms and the forms into which the work should be put. The table at the end of the book, or a separate table, is now used. We note that since there are no logarithms of negative numbers, products (and quotients) of negative numbers are obtained as if they were all positive; the proper sign is then prefixed in the result.

Example 1. Find the product of 49.6×82.7.

Solution:

$$\begin{array}{rl} \log 49.6 = & 1.69548 \\ \log 82.7 = & \underline{1.91751} \\ & 3.61299 = \log 4101.9. \end{array}$$

Hence the product is 4101.9.

Example 2. Find the value of $\dfrac{14.64 \times 811.2}{61.34}$

Solution:

$$\begin{array}{rl} \log 14.64 = & 1.16554 \\ \log 811.2 = & \underline{2.90913} \\ & 4.07467 \\ \log 61.34 = & \underline{1.78774} \\ & 2.28693 = \log 193.61. \end{array}$$

Example 3. Find the value of $9.47^2 \times 1.19^2$.

Solution:

$$\begin{array}{rl} \log 9.47^2 = & 1.95270 \\ \log 1.19^2 = & \underline{.15110} \\ & 2.10380 = \log 127.00. \end{array}$$

Writing down $\log 9.47^2 = 1.95270$ involves multiplying the logarithm of 9.47 by 2. This should be done mentally and no numbers should be written down, except those shown here. Log 1.19^2 is found in the same manner.

Example 4. Find the value of

$$\frac{21.3^2 \times 64.7^3}{34.1^4}$$

Solution:

$$\begin{array}{rl} \log 21.3^2 = & 2.65676 \\ \log 64.7^3 = & \underline{5.43270} \\ & 8.08946 \\ \log 34.1^4 = & \underline{6.13100} \\ & 1.95846 \\ & = \log 90.878 \end{array}$$

Example 5. Find the value of

$$\sqrt{\frac{17.4 \times 81.6}{24.3^2}}$$

Solution:

$$\begin{array}{rl} \log 17.4 = & 1.24055 \\ \log 81.6 = & \underline{1.91169} \\ & 3.15224 \\ \log 24.3^2 = & \underline{2.77122} \\ & 2 \,|\, \underline{.38102} \\ & .19051 \\ & = \log 1.5506 \end{array}$$

EXERCISES

Find the values of the following correct to five significant figures:

1. 71.23×4.26
2. $8.6^2 \times 2.91^3$
3. $51.4 \times 63.1 \times 2.91$
4. $\dfrac{82.3 \times 61.7}{41.2^2}$
5. $126 \times 3.7^4 \times 38$
6. $.98^2 \times 6.1^3 \times 12$
7. $440 \times .034^2 \times 6$
8. $\dfrac{53 \times 262 \times 18}{26.4^3}$
9. $7.14 \times 29.8 \times \sqrt{69.3}$
10. $\sqrt[3]{845} \times 79.3 \times \sqrt{29.6}$
11. $31.6 \times \sqrt[4]{76.8} \times \sqrt{69.2}$
12. $\dfrac{93^2 \times \sqrt[3]{806} \times 41}{31.2^4}$

60. Logarithm of a reciprocal.—Since the logarithm of 1 is 0, the logarithm of a fraction whose numerator is unity may be obtained by subtracting the logarithm of the denominator from zero. Thus $\log 1/384 = \log 1 - \log 384 = 0 - 2.58433 = 7.41567 - 10$. In this manner the logarithm of the reciprocal of a number may be written down directly from the logarithm of the number itself. The result must be written from right to left as the subtraction proceeds.

61. Multiplying by a reciprocal.—Instead of dividing by a number, one may multiply by its reciprocal. Thus, instead of dividing by 384 one may multiply by 1/384. Hence, instead of subtracting log 384 (2.58433) we may add log 1/384 (7.41567 − 10).

62. Cologarithms.—The logarithm of the reciprocal of a number is called the **cologarithm** (written colog) of the number itself. The use of cologs is suggested above and is shown in the following examples.

Example 1. Find the value of

$$\frac{8.19 \times 7.48}{36.7}$$

Solution:

$$\begin{array}{rl} \log\ 8.19 = & 0.91328 \\ \log\ 7.48 = & 0.87390 \\ \text{colog}\ 36.7 = & \underline{8.43533 - 10} \\ & 10.22251 - 10 \\ & = \log 1.6692 \end{array}$$

Example 2. Find the value of

$$\frac{18.3^4}{17 \times 8.92 \times 4.26}$$

Solution:

$$\begin{array}{rl} \log 18.3^4 = & 5.04980 \\ \text{colog}\ 17 = & 8.76955 - 10 \\ \text{colog}\ 8.92 = & 9.04964 - 10 \\ \text{colog}\ 4.26 = & \underline{9.37059 - 10} \\ & 32.23958 - 30 \\ & = \log 173.61 \end{array}$$

To find the colog of a number such as 384, we may write log 384 on a separate slip of paper and then subtract from 0, or colog 384 may be written down directly as suggested above. Only colog 384 should appear in the computation.

63. The advantage of using cologs.—The chief advantage in the use of cologs appears when we attempt to compute by subtracting logarithms (as was done on page 41) instead of adding cologs.

In Example 1 we should have to add the first two logarithms and then subtract the last. This would necessitate the writing of one extra line of figures.

In Example 2 we should have to add the last three logarithms and then subtract this sum from the first. This, however, is balanced in part by the greater difficulty of finding cologs. The chief advantage in the use of cologs arises when the divisor or dividend is expressed as the product of two or more factors. In such cases the work is made more compact and intelligible by their use.

EXERCISES

GROUP A

Use cologarithms for all divisions.

1. $\dfrac{89.6}{39 \times 43.7}$
2. $\dfrac{683}{13.4 \times 3.71^2}$
3. $\dfrac{16.5 \times 76.4}{19.7 \times 14.8}$
4. $\dfrac{89.47 \times 7.8}{5.4 \times 7.34^2}$
5. $\dfrac{3.7 \times 4.9 \times 8.7}{7.3 \times 9.4 \times 7.8}$
6. $\dfrac{10.4 \times 39.7}{27.3 \times 19.8^2}$
7. $\dfrac{37.2 \times 8.91}{73.9 \times 40.1}$
8. $\dfrac{98.46 \times 71.42}{42 \times 67 \times 81}$
9. $\dfrac{\sqrt{37.8} \times \sqrt{95.7}}{74.6 \times 46.7}$
10. $\sqrt{\dfrac{45.6 \times 29}{77.6 \times 84}}$
11. $\dfrac{12.9 \times 9.78}{7.19 \times \sqrt{764}}$
12. $\dfrac{41.64 \times 71.8}{\sqrt{241} \times \sqrt{874}}$
13. $\sqrt{\dfrac{14.82 \times 39.2}{41.28 \times 29.3}}$
14. $\dfrac{71.92 \times 28.4}{19.27 \times 48.2}$
15. $\dfrac{71.4 \times 82.6}{41.7 \times 28.6}$
16. $\dfrac{79.4 \times 9.26}{94.7 \times 69.2}$
17. $\dfrac{35.7 \times 98.2}{.048 \times 9426}$
18. $\dfrac{49.3 \times 176.8}{.471 \times 9840}$

GROUP B

1. $\sqrt{\dfrac{31.9 \times 7.3}{6.4 \times 3.9^3}}$
2. $\dfrac{5.7 \times \sqrt{4.7}}{9.8 \times 5.3}$
3. $\dfrac{6.8 \times \sqrt{8.9}}{\sqrt{4.7 \times 37}}$
4. $\sqrt{\dfrac{65.4 \times 37.8}{4.56 \times 78.3}}$
5. $\dfrac{\sqrt[3]{6.54} \times 38.7}{54.6 \times 84.7}$
6. $\dfrac{\sqrt{6.58} \times \sqrt{41.7}}{\sqrt{81.7} \times \sqrt{8.7}}$
7. $\dfrac{\sqrt{12.3 \times 45.6}}{1.28 \times 7.41}$
8. $\dfrac{92.7 \times 34.1}{16.4 \times 27 \times 8}$
9. $\sqrt{\dfrac{64.1 \times 72 \times 6}{13 \times 47 \times 61}}$
10. $\dfrac{31 \times 74 \times 16.9}{14.6 \times 27 \times 12}$
11. $\dfrac{19.8 \times 72.4 \times 31.7}{98.1 \times 24.7 \times 17.3}$
12. $\dfrac{640 \times .0139}{.024 \times 71.6 \times .028}$
13. $\sqrt{\dfrac{46 \times 319 \times 47}{71.6 \times 34.7 \times 21}}$
14. $\dfrac{87.8 \times 93.2 \times 17}{14 \times 12 \times 8 \times 94}$
15. $\dfrac{37.8 \times 78.3}{83.7 \times 7.38}$
16. $\dfrac{34.6 \times 27.8}{45.3 \times 78.2}$
17. $\sqrt{\dfrac{49.8 \times 26.4 \times 17}{34.2 \times 71.5 \times 19}}$
18. $\sqrt{\dfrac{31.6 \times 71.7 \times 24}{4.82 \times 92.7 \times 84}}$

GROUP C

1. $\dfrac{38.42 \times 6.54}{74.78 \times 913}$
2. $\dfrac{7.76 \times 67.8}{\sqrt{89.3} \times \sqrt{874}}$
3. $\dfrac{38.62 \times 29.42}{73 \times .191 \times 871}$
4. $\dfrac{49.3 \times 34.8}{84.8 \times 2.64}$
5. $\sqrt[3]{\dfrac{13.4 \times 6.43}{3.41 \times 26.4}}$
6. $\dfrac{62.8 \times 26.8}{39 \times 76 \times 84}$
7. $\sqrt{\dfrac{78.2 \times 34.6}{86.4 \times 2.64}}$
8. $\dfrac{7.31 \times 49.6}{81 \times 42 \times 73}$
9. $\sqrt{\dfrac{67.72 \times 84.9}{7.14 \times 27 \times 43}}$
10. $\dfrac{\sqrt{71.4 \times 26.8}}{\sqrt[3]{93.8 \times 176}}$
11. $\dfrac{3876}{\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{8760}}$
12. $\dfrac{61.9 \times 847}{39 \times 8.7 \times 64.8}$
13. $\dfrac{74.8 \times 48.7}{32.1 \times 13.2}$
14. $\dfrac{\sqrt{89.4} \times \sqrt[3]{74.8}}{\sqrt{1240} \times \sqrt[3]{942}}$
15. $\dfrac{37.62 \times 47.8}{27.3 \times 14 \times 89}$
16. $\dfrac{.450 \times .0174}{.045 \times 87 \times .0072}$
17. $\dfrac{19.84 \times 64.7}{34.7 \times 26 \times 92}$
18. $\sqrt[3]{\dfrac{74.3 \times 29.17}{62.8 \times 47.6}}$

64. Logarithms of trigonometric functions.—Tables giving the logarithms of trigonometric functions are most convenient for computation, since addition and subtraction of such functions occur but seldom.

65. Explanation of tables.—The table on the opposite page gives log sin, log cos, log tan, log cot, for every minute of angle from 18° to 19° and also from 71° to 72°. In the table use is made of the fact that cofunctions of complementary angles are equal.

Thus, the columns read downward give the functions of the angles from 18° to 19°, while read upward they give the functions from 71° to 72°.

Since the sine and cosine of an angle are never greater than unity, characteristics of their logarithms are negative. In the table opposite they are in reality -1. Instead $9 - 10$ is used, the -10 being omitted in printing.

Since the tangent of an angle greater than 45° and the cotangent of one less than 45° are greater than unity, the characteristic is zero, or a positive integer. These are given in the table.

66. Use of tabular differences in interpolating.—The use of the table of proportional parts is best understood from examples.

Example 1. Find $\log\sin 18^\circ 17' 18''$.

Solution: From the table find log sin 18° 17′ and note that log sin 18° 18′ is 38 greater. We then add the differences corresponding to 10″ and 8″. In no case should more than five figures be used.

$$\begin{aligned} \log\sin 18^\circ 17' &= 9.49654 \quad - 10\\ \text{Diff. for } 10'' &= .00006\\ \text{Diff. for } 8'' &= \underline{.00005}\\ \log\sin 18^\circ 17' 18'' &= 9.49665 \quad - 10 \end{aligned}$$

To find the difference corresponding to 10″, 20″ up to 50″, take 10 times the difference for 1″, 2″, etc.

Example 2. Find the angle A if $\log\tan A = 0.47456$.

Solution: From the table we find $\log\tan 71^\circ 27' = 0.47422$, with a tabular difference of 42; while the given logarithm shows a difference of 34. From the table of proportional parts the difference 34 corresponds to 49″ (28 corresponds to 40″ and 6.3 to 9″). Hence $A = 71^\circ 27' 49''$.

EXERCISES

In Examples 1–12 find the logs and in Examples 13–18 find A.

1. $\log\sin 18^\circ 35'$.
2. $\log\cos 18^\circ 52'$.
3. $\log\tan 18^\circ 39'$.
4. $\log\cot 18^\circ 7'$.
5. $\log\sin 71^\circ 14'$.
6. $\log\cos 71^\circ 38'$.
7. $\log\tan 71^\circ 52'$.
8. $\log\cot 71^\circ 8'$.
9. $\log\sin 71^\circ 13' 25''$.
10. $\log\cos 18^\circ 7' 8''$.
11. $\log\tan 18^\circ 45' 19''$.
12. $\log\cot 18^\circ 57' 9''$.
13. $\log\sin A = 9.50389$.
14. $\log\tan A = 9.51863$.
15. $\log\cos A = 9.97744$.
16. $\log\cot A = 9.53216$.
17. $\log\sin A = 9.97612$.
18. $\log\cos A = 9.49971$.

Logarithms of Trigonometic Functions (18° to 19°)

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60
1	9.49 037	39	221	43	779	817	5	59
2	076	39	264	42	736	812	4	58
3	115	38	306	43	694	808	4	57
4	153	39	349	43	651	804	4	56
5	192	39	392	43	608	800	4	55
6	231	38	435	43	565	796	4	54
7	269	39	478	42	522	792	4	53
8	308	39	520	43	480	788	4	52
9	347	38	563	43	437	784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	50
11	424	38	648	43	352	775	4	49
12	462	38	691	43	309	771	4	48
13	500	39	734	42	266	767	4	47
14	539	38	776	43	224	763	4	46
15	577	38	819	42	181	759	5	45
16	615	39	861	42	139	754	4	44
17	654	38	903	43	097	750	4	43
18	692	38	946	42	054	746	4	42
19	730	38	988	43	012	742	4	41
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	40
21	806	38	073	42	927	734	5	39
22	844	38	115	42	885	729	4	38
23	882	38	157	43	843	725	4	37
24	920	38	200	42	800	721	4	36
25	958	38	242	42	758	717	4	35
26	996	38	284	42	716	713	5	34
27	9.50 034	38	326	42	674	708	4	33
28	072	38	368	42	632	704	4	32
29	110	38	410	42	590	700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30
31	185	38	494	42	506	691	4	29
32	223	38	536	42	464	687	4	28
33	261	37	578	42	422	683	4	27
34	298	38	620	41	380	679	5	26
35	336	38	661	42	339	674	4	25
36	374	37	703	42	297	670	4	24
37	411	38	745	42	255	666	4	23
38	449	37	787	42	213	662	5	22
39	486	37	829	41	171	657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20
41	561	37	912	41	088	649	4	19
42	598	37	953	42	047	645	5	18
43	635	38	995	42	005	640	4	17
44	673	37	9.53 037	41	0.46 963	636	4	16
45	710	37	078	42	922	632	4	15
46	747	37	120	41	880	628	5	14
47	784	37	161	41	839	623	4	13
48	821	37	202	42	798	619	4	12
49	858	38	244	41	756	615	5	11
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10
51	933	37	327	41	673	606	4	9
52	970	37	368	41	632	602	5	8
53	9.51 007	36	409	41	591	597	4	7
54	043	37	450	42	550	593	4	6
55	080	37	492	41	508	589	5	5
56	117	37	533	41	467	584	4	4
57	154	37	574	41	426	580	4	3
58	191	36	615	41	385	576	5	2
59	227	37	656	41	344	571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	43	42
1″	0.7	0.7
2″	1.4	1.4
3″	2.2	2.1
4″	2.9	2.8
5″	3.6	3.5
6″	4.3	4.2
7″	5.0	4.9
8″	5.7	5.6
9″	6.5	6.3

1′	41	39
1″	0.7	0.7
2″	1.4	1.3
3″	2.1	2.0
4″	2.7	2.6
5″	3.4	3.3
6″	4.1	3.9
7″	4.8	4.6
8″	5.5	5.2
9″	6.2	5.9

1′	38
1″	0.6
2″	1.3
3″	1.9
4″	2.5
5″	3.2
6″	3.8
7″	4.4
8″	5.1
9″	5.7

1′	37
1″	0.6
2″	1.2
3″	1.9
4″	2.5
5″	3.1
6″	3.7
7″	4.3
8″	4.9
9″	5.6

1′	36
1″	0.6
2″	1.2
3″	1.8
4″	2.4
5″	3.0
6″	3.6
7″	4.2
8″	4.8
9″	5.4

1′	4	5
1″	0.07	0.08
2″	0.13	0.17
3″	0.20	0.25
4″	0.27	0.33
5″	0.33	0.42
6″	0.40	0.50
7″	0.47	0.58
8″	0.53	0.67
9″	0.60	0.75

71°

EXERCISES IN THE USE OF TABLES

Group A

Find each of the following:

1. $84 \sin 54^\circ 18'$
2. $173 \cos 53^\circ 35'$
3. $264 \tan 31^\circ 17'$
4. $528 \cot 61^\circ 24'$
5. $\sin 35^\circ 17' \times \cos 18^\circ 40'$
6. $\sin 62^\circ 48' \times \tan 37^\circ 18'$
7. $\sin 13^\circ 41' \times \cos 73^\circ 14'$
8. $36 \times 84 \times \sin 31^\circ 36'$
9. $12 \times 48.9 \times \cos 37^\circ 14'$
10. $\dfrac{\sqrt{81} \times \sqrt[3]{3.9}}{3.7 \times 8.5}$
11. $\dfrac{\sqrt[3]{6.4} \times \sqrt{3.2}}{8.7 \times 93.4}$
12. $\sqrt[3]{17.4 \times 81.7 \times 26}$
13. $\dfrac{6.24 \times .83 \times 6.4}{31.9 \times 7.6}$
14. $\dfrac{19.3 \times 6.7}{3.12 \times .38 \times 4.6}$
15. $\sqrt{1.94 \times 0.78 \times .06}$
16. $\dfrac{9.41^2 \times 7.3^2}{5.28^2 \times 1.74^2}$
17. $\dfrac{17.4^2 \times \sqrt[3]{1.89}}{41.7 \times 9.18}$
18. $\dfrac{7.52 \times 3\sqrt{85}}{16 \times 8 \times 53^2}$
19. $\sqrt[4]{2.81 \times 1.82 \times 9.12}$
20. $\dfrac{4.71 \times 28.4}{19 \times 37 \times 64}$
21. $\dfrac{4.56^2 \times 3.76^2}{2.4^3 \times 3.2^2 \times 6.1^2}$
22. $\sqrt{\dfrac{9.8 \times 34 \times 63}{9.1 \times 43 \times 37}}$
23. $\dfrac{5.91 \times 8.37}{9.25 \times 17 \times 91}$

Group B

1. $89 \sin 38^\circ 15'$
2. $76 \cos 65^\circ 47'$
3. $142 \tan 13^\circ 12'$
4. $31 \cot 71^\circ 19'$
5. $45 \sin 17^\circ \tan 48^\circ$
6. $\sin 81^\circ 12' \times \cos 17^\circ 13'$
7. $\sin 37^\circ \tan 48^\circ$
8. $\sqrt{7} \cot 37^\circ 14'$
9. $\sqrt[3]{19.2} \tan 51^\circ 13'$
10. $\dfrac{24 \times 73}{\sin 38^\circ 51'}$
11. $\dfrac{64 \times \sqrt[3]{81}}{93 \times 87 \times 67}$
12. $\sqrt{39.1 \times 78.2 \times 32.6}$
13. $\dfrac{191.6 \times 831.5}{\sqrt{39} \times \sqrt{79.4}}$
14. $\sqrt[3]{1246 \times .0938}$
15. $\dfrac{93.7 \times 37.9}{84.2 \times 24.8}$
16. $\sqrt{293 \times 13.78 \times .042}$
17. $\dfrac{174.2 \times 49.3 \times 14}{61.6 \times 45 \times 71 \times 21}$
18. $\sqrt{34.8 \times 67 \times 93 \times 12}$
19. $\sqrt{29.2 \times .117 \times .087}$
20. $\dfrac{67 \times 82 \times 93}{\sqrt{8760 \times 4982}}$
21. $\sqrt{\dfrac{30 \times 28 \times 67 \times 42}{17 \times 13 \times 41 \times 11}}$
22. $\sqrt[3]{67.1 \times 92.3 \times .074}$

Group C

1. $74 \sin 67^\circ 38'$
2. $95 \cos 49^\circ 23'$
3. $84 \tan 76^\circ 13'$
4. $104 \cot 18^\circ 47'$
5. $\sin 38^\circ \cos 57^\circ$
6. $\sin 59^\circ \tan 28^\circ$
7. $\cos 74^\circ \cot 37^\circ$
8. $\sin 15^\circ 12' \times \cos 37^\circ 14'$
9. $\sqrt{41} \sin 42^\circ 18'$
10. $\sqrt[3]{149} \cos 13^\circ 49'$
11. $\sqrt[4]{85} \cot 39^\circ$
12. $\dfrac{65.9}{\tan 18^\circ 42'}$
13. $\dfrac{35.4 \times 81.7}{\sin 62^\circ 48'}$
14. $\sqrt{167.1 \times 76.11 \times 71.6}$
15. $\sqrt{\sin 41^\circ 40' \times \cos 19^\circ}$
16. $\sqrt{\dfrac{51.4 \times 7.35}{145 \times 37.5 \times 15.7}}$
17. $\dfrac{87.6 \times 75.8 \times 67.2}{7.92 \times 27.9 \times 92.7}$
18. $\dfrac{45.6 \times 167.8 \times 9.12}{22.8 \times 8.39 \times 4.56}$
19. $\sqrt{8.49 \times 58.9 \times 32.4}$
20. $\dfrac{47.6 \times 64.7 \times 32.9}{31.7 \times 50.4 \times 24.9}$
21. $\dfrac{13.7 \times 42.7 \times 3146}{41.2 \times 5.46 \times 19.8}$
22. $\sqrt[3]{8.52 \times 7.94 \times 50.7}$
23. $\dfrac{6.71 \times 3.45 \times .0107}{.14 \times 1.82 \times 2.417}$

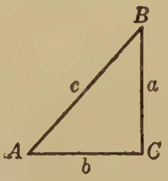
CHAPTER V

SOLUTION OF RIGHT TRIANGLES USING LOGARITHMS

WHEN one of the acute angles of a right triangle is known, the other acute angle may be found from the fact that these angles are complementary.

Thus, if in the right triangle, ABC, angle B is known, then angle A is found from the relation $\angle A = 90° - \angle B$.

It follows from this that in solving a right triangle, as ABC, only four parts, namely, a, b, c, and one acute angle as A, need be considered. Of these four parts, two must be given to make it possible to find the others.



67. The six problems in solving right triangles.—This gives rise to six problems, namely: find the remaining parts of the right triangle ABC, when any one of the following pairs of parts is given: $\angle A, a$; $\angle A, b$; $\angle A, c$; a, b; a, c; b, c.

Problem I. In a right triangle ABC, given $\angle A$ and side a, to find the remaining parts.

$$\textit{Formulas:}\begin{cases} \sin A = \dfrac{a}{c} \text{ and hence } c = \dfrac{a}{\sin A}. \\ \tan A = \dfrac{a}{b} \text{ and hence } b = \dfrac{a}{\tan A} = a \cot A. \end{cases}$$

Problem II. In a right triangle ABC, given $\angle A$ and side b, to find the remaining parts.

$$\textit{Formulas:}\begin{cases} \cos A = \dfrac{b}{c} \text{ and hence } c = \dfrac{b}{\cos A}. \\ \tan A = \dfrac{a}{b} \text{ and hence } a = b \tan A. \end{cases}$$

Problem III. Given $\angle A$ and side c, to find the remaining parts

Formulas: $a = c \sin A$ and $b = c \cos A$.

Problem IV. Given sides a, b, to find the remaining parts.

Formulas: $\tan A = \dfrac{a}{b}$ and $c = \dfrac{a}{\sin A}$

In solving this problem $\angle A$ is found first, and then $\sin A$ is found and substituted in the second formula.

The other two problems: *viz.*, when a, c or b, c are given, are solved in the same manner.

68. Logarithmic solutions.—The formulas used in the logarithmic solution of right triangles are exactly those used in solving by natural functions. The difference arises from the fact that logarithms are used in the computations. Proper arrangement of work is important in such computations, and the forms used in the solutions given below should be studied.

Example 1. In the right triangle, ABC, $\angle C = 90^\circ$, $\angle A = 42^\circ 23'$; $a = 18.3$. Find b and c.

Formulas: $b = \dfrac{a}{\tan A}$, $c = \dfrac{a}{\sin A}$.

$$\log b = \log a - \log \tan A. \qquad \log c = \log a - \log \sin A.$$

Forms:

$$\begin{array}{rl} \log 18.3 = & \\ \log \tan 42^\circ 23' = & \underline{\qquad\qquad} \\ \log b = & \\ b = & \end{array} \qquad \begin{array}{rl} \log 18.3 = & \\ \log \sin 42^\circ 23' = & \underline{\qquad\qquad} \\ \log c = & \\ c = & \end{array}$$

To facilitate the computation first write down the forms as shown above. Notice that 18.3 occurs in both forms and be sure to fill in both places at the same time. Similarly, tan 42° 23′ and sin 42° 23′ should be found at the same time, since this saves turning to this angle again.

When finished, the above forms are as follows:

$$\begin{array}{rl} \log 18.3 = & 11.26245 - 10 \\ \log \tan 42^\circ 23' = & \underline{\ 9.96028 - 10} \\ \log b = & 1.30217 \\ b = & 20.053 \end{array} \qquad \begin{array}{rl} \log 18.3 = & 11.26245 - 10 \\ \log \sin 42^\circ 23' = & \underline{\ 9.82872 - 10} \\ \log c = & 1.43373 \\ c = & 27.148 \end{array}$$

Example 2. Given $A = 56^\circ 14'$, $c = 23.76$, find a, and b.

Formulas: $a = c \sin A$ $\qquad b = c \cos A$

$$\begin{array}{rl} \log a = \log c + \log \sin A & \\ \log 23.76 = & 1.37585 \\ \log \sin 56^\circ 14' = & \underline{9.91976 - 10} \\ \log a = & 1.29561 \\ a = & 19.752 \end{array} \qquad \begin{array}{rl} \log b = \log c + \log \cos A & \\ \log 23.76 = & 1.37585 \\ \log \cos 56^\circ 14' = & \underline{9.74493 - 10} \\ \log b = & 1.12078 \\ b = & 13.206 \end{array}$$

Notice that log 23.76 occurs in both solutions. Also when finding log sin A in the form to the left fill in log cos A in the form to the right.

Example 3. Given $a = 31.62$, $b = 28.37$, find A, c.

Formulas:

$$\begin{array}{rl} \tan A = a/b & \\ \log \tan A = \log a - \log b & \\ \log 31.62 = & 1.49996 \\ \log 28.37 = & \underline{1.45286} \\ \log \tan A = & .04710 \\ A = & 48^\circ 6' 2'' \end{array} \qquad \begin{array}{rl} c = a/\sin A & \\ \log c = \log a - \log \sin A & \\ \log 31.62 = & 11.49996 - 10 \\ \log \sin A = & \underline{\ 9.87175 - 10} \\ \log c = & 1.62821 \\ c = & 42.481 \end{array}$$

Example 4. Given $a = 51.39$, and $c = 69.22$, find $\angle A$ and b.

Formulas:

$$
\begin{array}{rlrl}
\sin A &= a/c & b &= c \cos A \\
\log \sin A &= \log a - \log c & \log b &= \log c + \log \cos A \\
\log 51.39 &= 11.71088 - 10 & \log 69.22 &= 1.84023 \\
\log 69.22 &= \underline{1.84023} & \log \cos A &= \underline{9.82603 - 10} \\
\log \sin A &= 9.87065 - 10 & \log b &= 1.66626 \\
A &= 47^\circ\, 56'\, 16'' & b &= 46.372
\end{array}
$$

EXERCISES

Solve the right triangles ABC, in which C is 90°, and the given parts are as follows:

GROUP A

1. $A = 25^\circ\, 13'$ $\quad a = 51.67$
2. $A = 53^\circ\, 47'$ $\quad b = 15.64$
3. $A = 64^\circ\, 29'$ $\quad c = 36.45$
4. $a = 59.34$ $\quad b = 46.73$
5. $a = 97.43$ $\quad c = 108.42$
6. $c = 34.79$ $\quad b = 24.81$
7. $A = 17^\circ\, 49'$ $\quad a = 42.18$
8. $A = 71^\circ\, 27'$ $\quad b = 12.84$
9. $A = 27^\circ\, 17'$ $\quad c = 47.65$
10. $a = 57.31$ $\quad b = 13.75$
11. $a = 171.60$ $\quad c = 175.31$
12. $b = 61.17$ $\quad c = 153.17$
13. $A = 74^\circ\, 15'$ $\quad a = 27.64$
14. $A = 37^\circ\, 53'$ $\quad b = 42.67$
15. $A = 81^\circ\, 24'$ $\quad c = 62.74$
16. $a = 24.81$ $\quad b = 47.26$
17. $a = 14.82$ $\quad c = 27.46$
18. $b = 84.21$ $\quad c = 97.34$

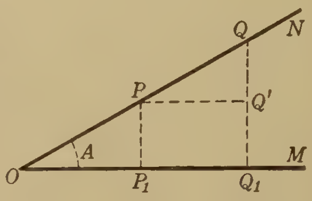
GROUP B

1. $A = 31^\circ\, 25'$ $\quad a = 17.56$
2. $A = 52^\circ\, 43'$ $\quad b = 75.61$
3. $A = 39^\circ\, 57'$ $\quad c = 94.17$
4. $a = 93.75$ $\quad b = 49.83$
5. $a = 207.64$ $\quad c = 342.14$
6. $b = 704.76$ $\quad c = 980.87$
7. $A = 7^\circ\, 59'$ $\quad a = 27.42$
8. $A = 19^\circ\, 31'$ $\quad b = 93.86$
9. $A = 47^\circ\, 48'$ $\quad c = 392.81$
10. $a = 79.34$ $\quad b = 91.62$
11. $a = 98.84$ $\quad c = 173.46$
12. $b = 76.61$ $\quad c = 107.89$
13. $A = 71^\circ\, 16'$ $\quad a = 47.64$
14. $A = 84^\circ\, 49'$ $\quad b = 136.47$
15. $A = 67^\circ\, 54'$ $\quad c = 337.94$
16. $a = 148.89$ $\quad b = 55.6$
17. $a = 418.53$ $\quad c = 640.8$
18. $b = 378.84$ $\quad c = 467.59$

GROUP C

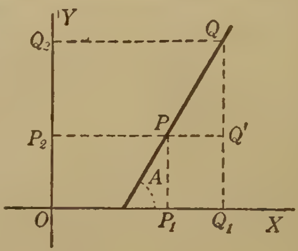
1. $A = 32^\circ\, 54'$ $\quad a = 24.80$
2. $A = 67^\circ\, 17'$ $\quad b = 482.61$
3. $A = 71^\circ\, 32'$ $\quad c = 284.16$
4. $a = 64.81$ $\quad b = 48.62$
5. $a = 91.76$ $\quad c = 142.63$
6. $b = 241.36$ $\quad c = 514.21$
7. $A = 81^\circ\, 13'$ $\quad a = 154.12$
8. $A = 74^\circ\, 19'$ $\quad b = 241.36$
9. $A = 63^\circ\, 51'$ $\quad c = 286.45$
10. $a = 45.39$ $\quad b = 53.49$
11. $a = 57.84$ $\quad c = 94.13$
12. $b = 74.93$ $\quad c = 101.54$
13. $A = 13^\circ\, 56'$ $\quad a = 41.51$
14. $A = 27^\circ\, 19'$ $\quad b = 54.11$
15. $A = 67^\circ\, 43'$ $\quad c = 85.83$
16. $a = 97.34$ $\quad b = 57.38$
17. $a = 191.45$ $\quad c = 275.83$
18. $b = 91.34$ $\quad c = 221.4$

69. Projection of one line on another.—Let two lines OM and ON meet in a point O, making an angle A. Let P and Q be two points on ON and let PP_1 and QQ_1 be perpendicular to OM. Then P_1Q_1 is called the projection of the line-segment PQ on the line OM. If PQ' is drawn parallel to OM, then $PQ' = P_1Q_1$ and $Q'PQ = \angle A$. Since $PQ' = PQ \cos \angle Q'PQ$, it follows that $P_1Q_1 = PQ \cos A$.



That is, *the projection upon a line of a segment lying in another line is equal to the projected segment multiplied by the cosine of the angle between the lines.*

70. Projection of a segment upon two perpendicular lines.—Let OX and OY be two lines at right angles to each other and P_1Q_1 and P_2Q_2 be the projections of PQ upon these two lines. Let A be the angle which the line PQ makes with OX. Then as above $P_1Q_1 = PQ \cos A$. Since $Q'Q = P_2Q_2$ it follows that $P_2Q_2 = PQ \sin A$. Hence we have the two equations,



$$P_1Q_1 = PQ \cos A, \quad P_2Q_2 = PQ \sin A$$

These equations are used frequently in applications to physics.

Example 1. Find the length of a segment if the projections on two perpendicular lines are a and b, respectively.

Solution: Let x = length of the segment, and let A be the angle which its line makes with one of the perpendicular lines.

Then, $$a = x \cos A \text{ and } b = x \sin A.$$

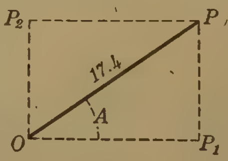
Squaring and adding,

$$a^2 + b^2 = x^2 \cos^2 A + x^2 \sin^2 A = x^2 (\cos^2 A + \sin^2 A) = x^2,$$

and $$x = \sqrt{a^2 + b^2}$$

Example 2. A ship sails at a rate of 17.4 knots (nautical miles per hour) along a straight course, making an angle A with an east-and-west line. At what rate is the ship going eastward and at what rate is it going northward?

Solution: If OP represents the distance the ship goes in one hour along its course, then OP_1 represents the distance it goes eastward and OP_2 the distance it goes northward. Hence its eastward speed is 17.4 cos A and its northward speed is 17.4 sin A.

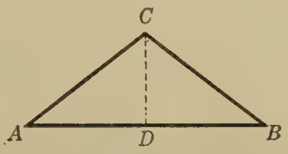


71. Solution of isosceles triangles.—Since an isosceles triangle may be divided into two right triangles by drawing the median upon its base, it follows that problems on isosceles triangles may be solved by means of the formulas already found.

Problem 1. Given one of the equal sides of an isosceles triangle and a base angle, to find the base and the altitude of the triangle.

Solution: Let ABC be the given triangle and CD the median upon the base. Then AD is half the base, and $CD = AC \sin A$, and $AD = AC \cos A$. Hence the altitude $= AC \sin A$ and the base $= 2\,AC \cos A$.

Thus, if $AC = 17.8$ and $A = 37° 30'$
then, $CD = 17.8 \sin 37° 30'$
and $AB = 2 \times 17.8 \cos 37° 30'$.



Problem 2. Given the sides of an isosceles triangle, to find its angles and the altitude on the base.

Solution: Using the figure of Example 1, we have $\cos A = \frac{AD}{AC}$. Since AD (half of AB) and AC are given, $\cos A$ may be found, and hence angle A may be found.

From the equation $DC = AC \sin A$ the altitude may then be found.

Thus, if $AC = 49.36$ and $AB = 43.64$

then, $\cos A = \frac{21.82}{49.36}$ and $A = 63° 46'$

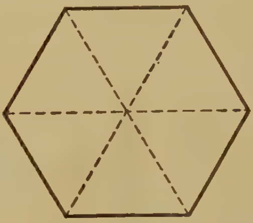
Hence, $DC = 49.36 \times \sin 63° 46'$

EXERCISES

Using the same figure as in Problem 1 above, find the required parts in each of the following.

	Given	*Required*
1.	$AC = 14.97$, $\angle A = 41° 32'$	AB, CD
2.	$AB = 43.12$, $CD = 26.46$	AC, $\angle A$
3.	$\angle A = 27° 34'$, $CD = 21.76$	AB, AC
4.	$\angle A = 35° 57'$, $AB = 187.8$	AC, CD
5.	$AC = 98.24$, $AB = 150.60$	$\angle A$, CD
6.	$\angle ACD = 49° 42'$, $AC = 216.3$	AD, DC
7.	$\angle ACD = 61° 34'$, $AD = 49.82$	AC, DC
8.	$\angle ACD = 58° 49'$, $CD = 73.67$	AD, AC
9.	$\angle ACB = 84° 40'$, $AD = 841.2$	AC, DC
10.	$\angle ACB = 92° 38'$, $CD = 67.81$	AD, AC
11.	$\angle ACB = 106° 46'$, $AC = 92.41$	AD, DC
12.	$\angle ACB = 128° 32'$, $AB = 73.84$	AC, DC

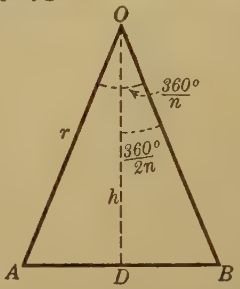
72. Solution of regular polygons.—Since a regular polygon may be divided into isosceles triangles by drawing lines from its center to its vertices, it follows that many problems on regular polygons may be solved by means of the formulas already found.



If a regular polygon has n sides, then the vertex angles of the isosceles triangles into which it is divided are all $360° \div n$.

Problem 1. Given the number of sides of a regular polygon and the length of each side, to find the radius of the circumscribed circle.

Solution: Let the number of sides be n. Let a be the length of each side, and r the radius of the circumscribed circle.



Then, $$\sin\frac{360°}{2n} = \sin\frac{180°}{n} = \frac{a}{2} \div r = \frac{a}{2r}$$

and $$r = \frac{a}{2\sin\frac{180°}{n}}.$$

Problem 2. A regular polygon of n sides is inscribed in a circle of radius r; find the length of each side of the polygon.

Solution: Using the same notation as in Problem 1,

$$r = \frac{a}{2\sin\frac{180°}{n}} \quad \text{and hence } a = 2r\sin\frac{180°}{n}.$$

Thus, if r equals unity, and $n = 16$, then $a = 2 \sin 11° 15'$.

Problem 3. Given the number of sides of a regular polygon and the length of each side, to find the area of the polygon.

Solution: Using the notation of Problem 1, let $OD = h$

Then, $$\frac{h}{a/2} = \frac{2h}{a} = \cot\frac{180°}{n} \text{ and } h = \frac{a}{2}\cot\frac{180°}{n}.$$

Hence, $$\text{area} = n \times \frac{a}{2} \times \frac{a}{2}\cot\frac{180°}{n} = n\frac{a^2}{4}\cot\frac{180°}{n}.$$

Thus, if $n = 12$, and $a = 20$, then area $= 12 \cdot \frac{400}{4} \cot\frac{180°}{12} = 1200 \cot 15°$.

EXERCISES

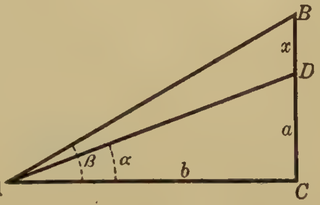
Using the notation of the above problems, find the required parts in each of the following:

Given	Required	Given	Required
1. $a = 6, n = 10$	r, h	**4.** $r = 8, n = 12$	a, area
2. $a = 15, n = 7$	r, h	**5.** $r = 20, n = 32$	a, area
3. $r = 10, n = 9$	a, h	**6.** $a = 2, n = 24$	r, area

73. The solution of problems.—In solving problems by using the formulas of trigonometry it is important to obtain a general algebraic solution before doing any of the work of numerical computation. This is illustrated in the solutions of problems given on this page.

Problem 1. In the figure find the line x (DB) in terms of α, β, and b.

Solution: Let $CD = a$.



Then, $\tan\alpha = \dfrac{a}{b}$

and $\tan\beta = \dfrac{a+x}{b}$.

Hence, $$\tan\beta - \tan\alpha = \frac{a+x}{b} - \frac{a}{b} = \frac{x}{b}$$

and $$x = b\,(\tan\beta - \tan\alpha).$$

Thus, if

$\beta = 37^\circ\,20'$, $\alpha = 21^\circ\,40'$,

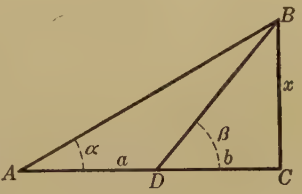
and $b = 140$,

then, $b\,(\tan\beta - \tan\alpha) = 140\,(.7627 - .3973) = 51.156 = x$.

In this case the table of natural functions should be used to find $\tan\beta - \tan\alpha$.

Problem 2. In the figure find x in terms of a, α, and β.

Solution: Let $DC = b$.



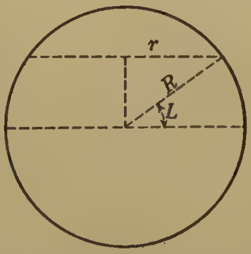
Then, $$\cot\beta = \frac{b}{x}, \text{ and } \cot\alpha = \frac{b+a}{x}.$$

Hence, $$\cot\alpha - \cot\beta = \frac{b+a}{x} - \frac{b}{x} = \frac{a}{x}$$

and $$x = \frac{a}{\cot\alpha - \cot\beta}.$$

Thus, if $\alpha = 14^\circ\,50'$, $\beta = 31^\circ\,30'$, and $a = 500$,

then, $$x = \frac{500}{3.7760 - 1.6319} = \frac{500}{2.1441} = 233.20.$$



Problem 3. Find the length of a degree of longitude on the earth when the latitude and the radius of the earth are given.

Solution: Let L = latitude in degrees, R the radius of the earth (it is assumed that the earth is spherical in shape), and r the radius of the circle of latitude. Then, $r = R\cos L$.

Hence the length of a degree of longitude is equal to the length of a degree on the equator multiplied by the cosine of the latitude.

Thus the length of a degree of longitude on the 42d parallel is cos 42° times the length of one degree at the equator.

PROBLEMS: GROUP A

1. A ladder 32 feet long is placed against a building so as to make an angle of 70° with the ground. How far from the building is the lower end of the ladder?

2. A line joins a vertex of a square to the middle point of a side. What angles does this line make with the three sides of the square which it meets?

3. A segment AB is projected on a line l into a segment A_1B_1. What is the length of AB if $A_1B_1 = 21.7$ and the angle between the lines AB and l is $19° 15'$?

4. An airplane flying at a rate of 115 miles an hour is making an angle of $51° 30'$ with an east-and-west line. How fast is it moving eastward? How fast is it moving northward?

5. Find the altitude of an isosceles triangle whose base is 12 and whose base angles are $43° 40'$.

6. Find the angles of an isosceles triangle whose equal sides are 28 and whose base is 18.

7. Find the length of a side of a regular polygon of 12 sides inscribed in a circle whose radius is 8.

8. A side of a regular polygon of 16 sides is 3.7. Find the radius of the circumscribed circle.

9. Find the area of a regular polygon of 12 sides inscribed in a circle whose radius is 8.

10. Find the length of a degree of longitude at latitude $37° 30'$, the radius of the earth being 3956 miles.

11. The blades of a pair of shears are $6\frac{1}{2}$ inches long from the screw to the points. What is the angle formed by the blades when the points are $4\frac{1}{2}$ inches apart?

12. The diagonals of a regular inscribed pentagon are 14.8. Find the radius of the circumscribed circle.

GROUP B

1. The top of a cliff is known to be 270 feet above the level of a lake. The angle of elevation of the cliff from a boat on the lake is $17° 45'$. What is the horizontal distance from the boat to the cliff?

2. A rectangle is five times as long as it is wide. What angles does a diagonal of this rectangle make with its sides?

3. A line segment 17 inches long makes an angle of $21° 50'$ with a line AB. Find the length of the projection of this segment on the line AB.

4. A river flows directly southward at a rate of 6 miles per hour. A boat pointed straight across the river goes westward at a rate of 12 miles per hour. What is the actual direction of the line in which the boat travels?

5. Find the angles of an isosceles triangle whose base is 16 and whose altitude is 10.

6. Find the sides of an isosceles triangle whose base angles are $48° 10'$ and whose altitude is 18.

7. Find the length of a side of a regular polygon of 10 sides inscribed in a circle of radius 12.

8. Find the radius of a circle circumscribed about a regular polygon of 7 sides if its sides are 9.

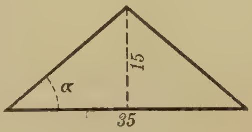
9. Find the area of a regular ten-sided polygon inscribed in a circle whose radius is 8.

10. A railway rises 150 feet in going one mile horizontal distance. What angle does the track make with a horizontal line?

11. How far does a person living in latitude 43° 15′ travel in one hour on account of the earth's rotation?

12. The diagonals of a regular pentagon are 21.7. Find the radius of its inscribed circle.

Group C



1. A house 35 feet in width is to be provided with a gable roof such that the peak is 15 feet above the eaves. What is the pitch of the roof? (Find tan α.)

2. Of the two legs of a right triangle one is three times the other. Find the angles which the hypotenuse makes with the legs of the triangle.

3. A line segment 12 inches long is projected into a segment 8.5 inches long on a line AB. Find the angle between the projected segment (extended if necessary) and the line AB.

4. A ship is sailing at the rate of 21 miles per hour in a direction of 64° 40′ north of east. At what rate is the ship going northward?

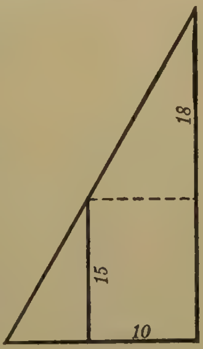
5. Find the angles of an isosceles triangle whose sides are 20, 30, and 30.

6. Find the base of an isosceles triangle whose base angles are 38° 20′ and whose equal sides are 24.

7. Find the radius of a circle circumscribed about a regular polygon of 14 sides of length 7.

8. Find the length of a side of a regular ten-sided polygon inscribed in a circle of radius 12.

9. Find the area of a regular polygon of 24 sides inscribed in a circle whose radius is 12.



10. A wire stay on an electric-light pole is fastened on the pole 12.3 feet from the ground, and in the ground 13.5 feet from the pole. What angle does the stay make with the ground?

11. A wall is 15 feet high and 10 feet from a house. What must be the length of a fire ladder which is to reach a window 18 feet above the top of the wall and just touch the wall?

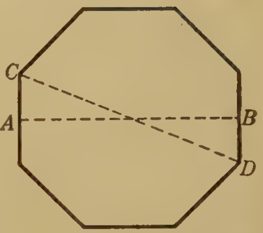
12. Two parallel chords on the same side of the center of a circle are 3 and 5 inches distant from the center of the circle. Find the number of degrees of arc intercepted between these chords if the radius of the circle is 8.

Miscellaneous Problems

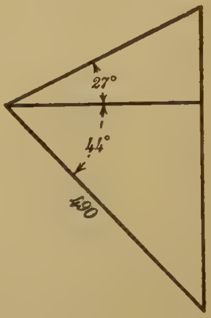
1. Find to the nearest degree the angle of elevation of an inclined plane which rises 4 feet 5 inches in a horizontal distance of 14 feet.

2. A regular octagonal nut measures $1\frac{7}{8}$ inches from A to B. Find the distance from C to D.

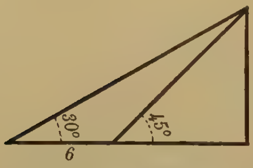
3. The beam of a hoisting crane is 36 feet long. How high does it reach when making an angle of 71° with the ground?

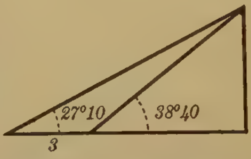


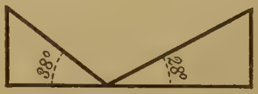
4. In the side of a hill which slopes upward at an angle of 27° a tunnel is bored, sloping downward at an angle of 44° with the horizontal. What is the vertical distance to the surface of the hill from a point 490 feet down the tunnel?



5. The shadow of a post is 6 feet longer when the sun's altitude is 30° than when it is 45°. Find the height of the post.







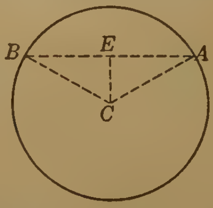
6. A ship sailing on a straight course observed a lighthouse at an angle of 27° 10′ to the left of its course. After sailing 3 miles this angle is 38° 40′. What is the shortest distance from the lighthouse to the course of the ship?

7. Two pillars of equal height stand on opposite sides of a roadway which is 100 feet wide. At a point in the road between the pillars, the angles of elevation of their tops are 28° and 38°, respectively. Find the height of the pillars and the position of the point of observation.

8. Two towers are 75 and 225 feet high, respectively. From the foot of the second, the elevation of the top of the first is found to be 60°. Find the elevation of the top of the second from the foot of the first.

9. If in the figure $AC = 6$, and $CE = 4$, find the area of that part of the circle which is above the line AB.

Suggestion: (1) Find $\angle ACB$, (2) find area of sector ACB, (3) find area of triangle ACB.



10. Find the angle between a diagonal of a cube and an edge of the cube which meets it.

11. A pyramid whose base is a square with sides 14 has an altitude 10. Find the angle between an edge and a side of the base meeting it.

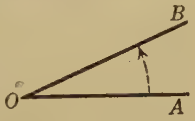
CHAPTER VI

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

In order to solve obtuse triangles it is necessary to extend the definitions of the trigonometric functions so as to make them apply to obtuse angles. Later we shall find uses for functions of any angles whatever.

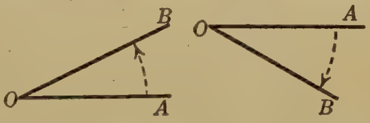
74. Generation of angles.—An angle may be regarded as *generated* by making one of its sides rotate about the vertex while the other side remains fixed. The angle may be considered as the amount of rotation of a line about a fixed point in the line.

Thus, the side OA may be regarded as fixed, while OB rotates about the point O as a pivot. In this manner the side OB may continue to rotate indefinitely, sweeping out any number of degrees whatever.

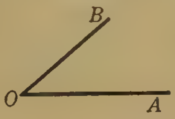


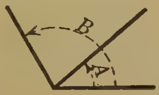
75. Initial and terminal sides of an angle.—The fixed side OA of the angle AOB is called the **initial side** and the side OB, which generates the angle, is called the **terminal side.**

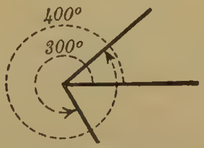
76. Positive and negative angles.—When the side OB swings in the counter-clockwise direction, the angle is regarded as positive, and when OB swings in the clockwise direction the angle is negative.



77. Notation for angles.—As in plane geometry, an angle is denoted by a letter at the vertex or by three letters, one on each side and one at the vertex. Frequently an angle is denoted by the number of degrees it contains. This is especially convenient for angles greater than a straight angle. Sometimes an arc within the angle is used to indicate it.



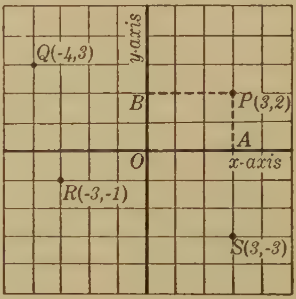




In the sequel we shall consider angles even greater than 360°. Thus we shall use expressions like sin 480°, tan 390°, etc. The definitions of trigonometric functions given in this chapter apply to all such angles and also to negative angles.

78. Coördinate axes.—Before defining the functions of general angles, it is necessary to describe a special method of locating points in a plane.

Two fixed lines at right angles to each other are taken as reference lines and the distances and directions from these lines to a point serve to locate the point. The fixed lines are called the **coördinate axes,** the horizontal axis being called the **x-axis,** and the vertical axis the **y-axis.**



79. Coördinates.—The distances AP and BP from the x-axis and y-axis, respectively, to the point P are called the **coördinates** of the point P. The distance BP from the y-axis (parallel to the x-axis) is called the **abscissa** of P, and the distance AP from the x-axis is the **ordinate** of P.

In giving the coördinates of a point the abscissa is always given first. Thus, the point (3, 2) is the point whose abscissa is 3 and whose ordinate is 2. The abscissa is often denoted by x and the ordinate by y. The point (3, 2) is the point $x = 3$, $y = 2$.

80. Positive and negative coördinates.—If the point P (or S) is to the right of the y-axis, its abscissa is positive, and if Q (or R) is to the left of the y-axis the abscissa is negative. If P (or Q) is above the x-axis, its ordinate is positive, and if R (or S) is below the x-axis, its ordinate is negative.

This method of locating a point in a plane is closely analogous to the method used in geography for locating a point on the earth's surface. The latitude is the distance in degrees north or south of the equator and the longitude is the distance east or west of the prime meridian (the meridian of Greenwich).

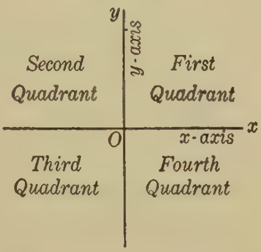
The horizontal axis then corresponds to the equator and the vertical axis to the prime meridian. Instead of *north* and *south, east* and *west* used in geography to indicate direction from the reference lines, the signs + and − are used.

Hence, both in geography and in mathematics, two numbers giving the distances and directions from two fundamental lines serve to locate a point.

In the United States land survey the description of location of townships also uses this method. Principal meridians are established and also parallels of latitude. A township is then said to be so far east or west of a particular principal meridian, and so far north or south of a certain parallel.

In this case the parallel of latitude corresponds to the x-axis and the principal meridian corresponds to the y-axis.

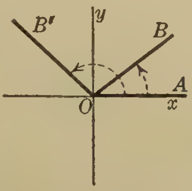
81. Quadrants.—The coördinate axes divide the plane into four parts called **quadrants.** These quadrants are called the first, second, third, and fourth quadrants, as indicated in the figure.

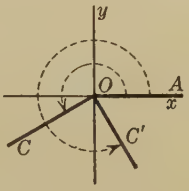


82. Angles in the various quadrants.—In studying angles, the vertex is placed at the intersection point O of a pair of coördinate axes, with the initial side extending to the right along the x-axis.

If the terminal side of an angle so placed falls within the first quadrant the angle is said to be in the first quadrant. That is, an angle between 0° and 90° is in the first quadrant.

If the terminal side of the angle lies within the second quadrant, the angle is in the second quadrant. That is, angles between 90° and 180° are in the second quadrant. Thus AOB is in the first quadrant and AOB' is in the second.





Similarly, the angles AOC and AOC' in the second figure are said to be in the third and fourth quadrants, respectively.

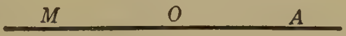
That is, angles between 180° and 270° are in the third quadrant and angles between 270° and 360° are in the fourth quadrant.

In general, any angle is said to be in the quadrant in which its terminal side lies.

This statement applies also to negative angles. Thus, an angle between $-270°$ and $-360°$ is in the first quadrant, and an angle between $-180°$ and $-270°$ is in the second quadrant, and so on.

83. Positive and negative line segments.—A line-segment on the x-axis is positive if read from left to right, no matter whether the segment lies to the right or the left of the origin.

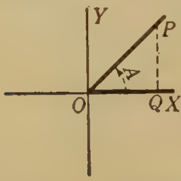
Thus, OA and MO are positive segments, while AO and OM are negative.

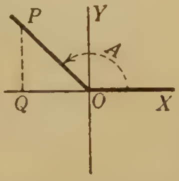


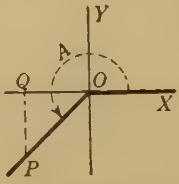
Similarly, segments read upward are positive and those read downward are negative.

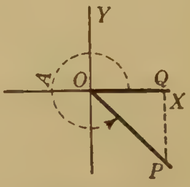
Any segment read in the direction away from the origin on the terminal side of an angle is positive; such a segment on a prolongation of the terminal side backward through the origin is negative.

84. Definitions of functions of any angle.—Consider a given angle A of any size and quality. Place the angle upon a set of coordinate axes XOY (any one of the four figures below), with its vertex at O and its initial side along OX. On its terminal side select any point P. From P drop a perpendicular QP to the x-axis.









We have the following definitions applicable to all four figures:

$$\sin A = \frac{QP}{OP} = \frac{\text{ordinate}}{\text{distance}} \qquad \csc A = \frac{OP}{QP} = \frac{\text{distance}}{\text{ordinate}}$$

$$\cos A = \frac{OQ}{OP} = \frac{\text{abscissa}}{\text{distance}} \qquad \sec A = \frac{OP}{OQ} = \frac{\text{distance}}{\text{abscissa}}$$

$$\tan A = \frac{QP}{OQ} = \frac{\text{ordinate}}{\text{abscissa}} \qquad \cot A = \frac{OQ}{QP} = \frac{\text{abscissa}}{\text{ordinate}}$$

It is clear that for acute angles these definitions are identical with those already given.

SIGHT WORK

1. Formulate for each of the other functions a definition similar to the following: "The sine of any angle is the ratio of the ordinate of any point on its terminal side to the distance of that point from the vertex."

2. If angle QOP is in the first quadrant, is the abscissa of P positive or negative? Is the ordinate positive or negative in the second quadrant?

3. Remembering that OP measured along the terminal side of the angle is positive, is $\sin A$ positive or negative in the second quadrant? in the third? Is $\cos A$ positive or negative in these quadrants? Answer similar questions for tangent, cotangent, secant, cosecant.

85. Coterminal angles and their functions.—Angles whose initial and terminal sides coincide are said to be **coterminal.** It is clear that if 360° is added to the angle A in the above figure, or subtracted from it, the resulting angle will be coterminal with A. Similarly, if twice 360° ($2 \cdot 360°$), $3 \cdot 360°$, etc., is added to A, or subtracted from it, the resulting angle will be coterminal with A.

Again, in each of these figures there is a negative angle less than 360° which is coterminal with A.

It follows at once from the definitions of trigonometric functions that any function is the same for all *coterminal angles*.

86. Relations between trigonometric functions.—It is easy to show that the relations (1)—(8) given on page 14 hold for the functions of any angle as defined on page 60.

The relations (1) $\sin A = 1/\csc A$, (2) $\cos A = 1/\sec A$, (3) $\tan A = 1/\cot A$, (4) $\tan A = \dfrac{\sin A}{\cos A}$, (5) $\cot A = \dfrac{\cos A}{\sin A}$ follow immediately from the definitions as in §§13, 14.

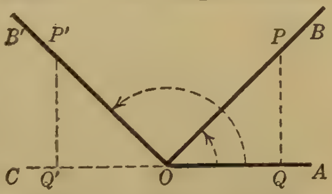
To show that (6) $\sin^2 A + \cos^2 A = 1$, (7) $\tan^2 A + 1 = \sec^2 A$, (8) $\cot^2 + 1 = \csc^2 A$ hold for any angle, we need only to notice that in all the figures on page 60

$$\overline{OQ}^2 + \overline{QP}^2 = \overline{OP}^2$$

and then to proceed as in §15.

The table of relations among the functions given on page 16 was derived by means of these eight relations. Hence that table has now been shown to hold for the functions of any angle. It also follows that the examples and exercises on pages 16–18 in the reduction of trigonometric identities, are exactly the same for functions of any angle as for functions of an acute angle.

87. Functions of supplementary angles.—Two angles whose sum is 180° (two *rt* ∠*s*) are said to be **supplementary.** In the figure let ∠AOB and ∠AOB' be supplementary. Lay off $OP = OP'$, and construct PQ and $P'Q'$ perpendicular to the X-axis. It is clear that ∠AOB = ∠$Q'OB'$, and hence that $QP = Q'P'$ and $OQ = -OQ'$. It follows at once that,



(1) $\sin AOB = \sin AOB'$,
(2) $\cos AOB = -\cos AOB'$,
(3) $\tan AOB = -\tan AOB'$,
(4) $\cot AOB = -\cot AOB'$,
(5) $\sec AOB = -\sec AOB'$,
(6) $\csc AOB = \csc AOB'$.

These propositions hold also for negative angles. This subject is considered further in Chapter IX. Stated in words, these formulas are:

If two angles are supplementary their sines are equal, and also their cosecants. The cosines, tangents, cotangents, and secants of these angles are equal numerically, but opposite in sign.

Since A and $180° - A$ are supplementary angles, this rule may be stated as follows:

$\sin A = \sin(180° - A)$ [12] $\qquad$ $\cos A = -\cos(180° - A)$ [13]
$\tan A = -\tan(180° - A)$ [14] $\qquad$ $\cot A = -\cot(180° - A)$ [15]
$\sec A = -\sec(180° - A)$ [16] $\qquad$ $\csc A = \csc(180° - A)$ [17]

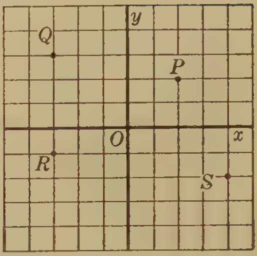
SIGHT WORK

1. What is meant by coördinate axes?

2. What is meant by the coördinates of a point?

3. Describe the method used in geography for locating a point on the earth's surface. State fully how this corresponds to the method of locating a point by its coördinates.

4. In the figure what are the coördinates of the point P? of Q? of R? of S?



5. Rule a sheet of paper as in the figure of Example 4 and locate the points whose coördinates are 1, 2; 3, 1; – 2, 1; – 1, – 2; – 2, 1; 4, – 1.

6. What is the sign of the abscissa of a point in the first quadrant? in the second quadrant? in the third quadrant? in the fourth quadrant?

7. What is the sign of the cosine of an angle in the first quadrant? in the second quadrant? in the third quadrant? in the fourth quadrant?

8. What is the sign of the ordinate of a point in the first quadrant? in the second quadrant? in the third quadrant? in the fourth quadrant?

9. What is the sign of the sine of an angle in the first quadrant? in the second quadrant? in the third quadrant? in the fourth quadrant?

10. In what quadrants do the ordinate and abscissa of a point have the same sign?

11. In what quadrants are the tangent and cotangent of an angle positive? In what quadrants are they negative?

12. How are the signs of the sine and cosecant of an angle related? of the cosine and secant? of the tangent and cotangent?

13. What angle is the supplement of 45°? of 60°? of 100°? of 120°? of 150°? of 170°?

14. Give the following in terms of functions of an angle in the first quadrant: sin 145°, cos 145°, tan 145°, cot 145°, sec 145°, cosec 145°.

15. Give all the functions of 165° in terms of functions of an angle in the first quadrant.

16. Find log sin 115°.

Suggestion: $\sin 115° = \sin(180° - 115°) = \sin 65°$. Hence find log sin 65°.

17. Find the logarithm of the absolute value of cos 125°.

Suggestion: $\cos 125° = -\cos(180° - 125°) = -\cos 55°$. Hence find log cos 55°.

18. Find the logarithms of the absolute values of the following: tan 142°, cot 134°, sec 98°, csc 104°.

19. State how you would find $\frac{\sin 94° \cos 142°}{\tan 110° \cot 130°}$ using logarithms.

Suggestion: (a) How would you find the absolute value of this expression? (b) How would you find its sign?

CHAPTER VII

FUNCTIONS OF SUMS AND DIFFERENCES OF ANGLES

In this chapter certain additional formulas are developed which are needed in the general solution of the triangle, a more general treatment of the relations of the trigonometric functions of angles being reserved for Chapter IX.

From the tables of natural functions it is apparent that a change in a function of an angle is not proportional to the corresponding change in the angle.

Thus, $\sin 20° = .3420$ and $\sin 40° = .6428$,
and hence, $\sin 20° + \sin 40° = .9848$, while
$\sin (20° + 40°)$, or $\sin 60°$, $= .8660$.

We now proceed to develop formulas giving the functions of the sum or difference of two angles in terms of the functions of the angles themselves. The fundamental formulas are:

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \qquad [18]$$
$$\sin (A - B) = \sin A \cos B - \cos A \sin B \qquad [19]$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B \qquad [20]$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B \qquad [21]$$

From these, other formulas will be derived giving the tangent and cotangent of the sum and of the difference of two angles.

88. Formulas used in this chapter.—In deriving these formulas use will be made of some that have already been developed. Among these the more important are:

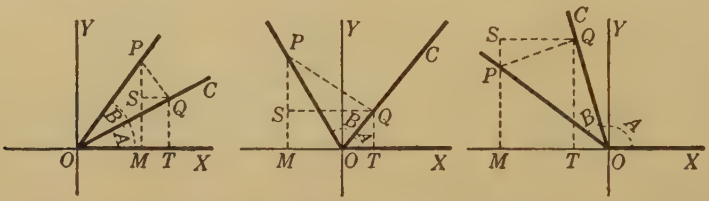
(1) Using the usual notation for the parts of a right triangle, $a = c \sin A$, $b = c \cos A$ (§§22, 23).

(2) If two angles A and B are supplementary, then we have $\sin A = \sin B$ and $\cos A = -\cos B$ (§87).

We notice that the sum of any two angles of a triangle is less than 180°, and that if one such angle is obtuse, the other is acute. We also recall from plane geometry that if the sides of $\angle A$ are perpendicular, respectively, to the sides of $\angle B$, then angles A and B are equal or supplementary.

In this chapter we will confine ourselves to positive angles and hence the difference of two angles will be taken in such a way as to make it positive.

89. Formula for the sine of the sum of two angles.—For the present we will consider the case of two angles, A and B, each of which is positive and whose sum is less than 180°. In each of the three figures, let $\angle XOC$ be the angle A, and $\angle COP$ be the angle B.



Notice that the initial side of angle B coincides with the terminal side of angle A. Then $\angle XOP$ is the angle $A + B$.

From P, a point in the terminal side of $A + B$, draw $MP \perp OX$ and $PQ \perp OC$. From Q draw $TQ \perp OX$ and $QS \perp MP$ (extended if necessary, as in the third figure).

In the $\triangle SPQ$, $SP \perp OX$, $PQ \perp OC$, and hence $\angle SPQ = \angle A$, or $\angle SPQ$ and $\angle A$ are supplementary.

Then for either of the first two figures, we have:

$$\begin{aligned} TQ &= OQ \sin A & SP &= QP \cos A \\ OT &= OQ \cos A & QP &= OP \sin B \\ SQ &= QP \sin A & OQ &= OP \cos B. \end{aligned}$$

Hence,

$$\begin{aligned} \sin(A+B) &= \frac{MP}{OP} = \frac{TQ+SP}{OP} = \frac{OQ \sin A + QP \cos A}{OP} \\ &= \frac{OP \cos B \sin A + OP \sin B \cos A}{OP} \\ &= \cos B \sin A + \sin B \cos A. \end{aligned}$$

That is,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

In applying this proof to the third figure notice that $MP = TQ - SP$ and also that $\angle SPQ$ and $\angle A$ are supplementary.

Stated in words this formula is: *the sine of the sum of two angles is equal to the sine of the first times the cosine of the second plus the cosine of the first times the sine of the second.*

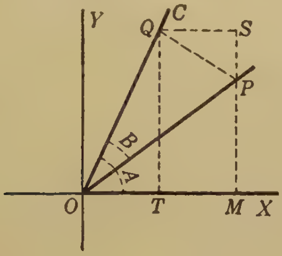
90. The cosine of the sum of two angles.—From the figures of §89 and the values of the line segments given there we have

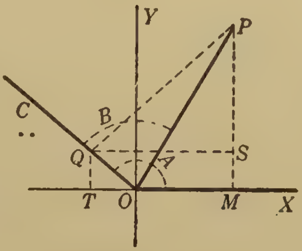
$$\cos(A+B) = \frac{OM}{OP} = \frac{OT - SQ}{OP} = \frac{OQ \cos A - QP \sin A}{OP}$$
$$= \frac{OP \cos B \cos A - OP \sin B \sin A}{OP}.$$

That is,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

91. The sine of the difference of two angles.—In the case when $\angle A$ is acute (the first figure) we have





$$\sin(A-B) = \frac{MP}{OP} = \frac{TQ - SP}{OP}$$

Using the values of TQ, OT, SQ, SP, QP, OQ given in §89, we have $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

In the case when A is obtuse (the second figure) we have

$$\sin(A-B) = \frac{MP}{OP} = \frac{TQ + SP}{OP}.$$

Using the same substitutions as in the first case and noting that $\angle A$ and $\angle SPQ$ are supplementary and that hence $\cos \angle SPQ = -\cos A$, we obtain the same formula.

92. The cosine of the difference of two angles.—In the case when $\angle A$ is acute (the first figure in §91), we have

$$\cos(A-B) = \frac{OM}{OP} = \frac{OT + QS}{OP}.$$

Making the same substitution as above we have

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

In the case when $\angle A$ is obtuse (the second figure in §91) we have

$$\cos(A-B) = \frac{OM}{OP} = \frac{QS - TO}{OP}.$$

Using the same substitution as above, except that $TO = -OQ \cos A$ (TO is positive while $\cos A$ is negative), we have the same formula.

93. The sine and cosine of double-angles and half-angles.— By letting $B = A$ in the formulas for $\sin (A + B)$ and $\cos (A + B)$ we have

$$\sin 2A = 2 \sin A \cos A \quad [22] \quad \text{and} \quad \cos 2A = \cos^2 A - \sin^2 A \quad [23]$$

If in these formulas $2A$ is replaced by P, we have

$$\sin P = 2 \sin \frac{P}{2} \cos \frac{P}{2} \text{ and } \cos P = \cos^2 \frac{P}{2} - \sin^2 \frac{P}{2}.$$

Using [9] we have,

$$\cos P = \cos^2 \frac{P}{2} - \sin^2 \frac{P}{2} = 2 \cos^2 \frac{P}{2} - 1 = 1 - 2 \sin^2 \frac{P}{2},$$

and hence

$$\sin^2 \frac{P}{2} = \frac{1 - \cos P}{2} \quad \text{or} \quad \sin \frac{P}{2} = \pm \sqrt{\frac{1 - \cos P}{2}} \qquad [24]$$

and

$$\cos^2 \frac{P}{2} = \frac{1 + \cos P}{2} \quad \text{or} \quad \cos \frac{P}{2} = \pm \sqrt{\frac{1 + \cos P}{2}} \qquad [25]$$

EXERCISES

1. Using the formula for $\sin (A + B)$ find $\sin 75°$ $(75° = 45° + 30°)$.

2. By means of the formula for $\cos (A - B)$ and using the combination $30° - 30° = 0$ verify that $\cos 0° = 1$.

3. If $\sin A = 3/4$ and $\sin B = 1/3$, find $\sin (A + B)$ and also $\sin (A - B)$. *Suggestion:* First find $\cos A$ and $\cos B$.

4. If $\cos A = 1/3$, $\cos B = 2/3$, find $\cos (A + B)$ and also $\cos (A - B)$.

5. If $\sin A = 1/2$ and $\cos B = 1/3$, find $\sin (A + B)$.

6. If $\cos A = 1/3$ and $\sin B = 3/4$, find $\cos (A - B)$.

Prove the following:

7. $\sin (45° - A) \cos (45° - B) - \cos (45° - A) \sin (45° - B) = \sin (B - A)$. *Suggestion:* What is the sine of the difference of the two angles $45° - A$ and $45° - B$?

8. $\cos A \cos (B - A) - \sin A \sin (B - A) = \cos B$.

9. $\sin (m + 1) A \cos (m - 1) A - \cos (m + 1) A \sin (m - 1) A = \sin 2A$.

10. $\cos (m + 1) A \cos (m - 1) A + \sin (m + 1) A \sin (m - 1) A = \cos 2A$.

11. $\cos (A + B) \cos (A - B) = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A - \sin^2 B$.

12. $\sin (60° - A) \cos (60° - B) - \cos (60° - A) \sin (60° - B) = \sin (B - A)$. *Suggestion:* What is the difference of the angles $60° - A$ and $60° - B$?

13. $\cos (B - A) \cos B + \sin (B - A) \sin B = \cos (- A)$.

14. $\sin (m + 2) A \cos (m - 2) A - \cos (m + 2) A \sin (m - 2) A = \sin 4A$.

94. Formula for tan (A + B).—Since $\tan A = \sin A/\cos A$, it follows that

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)}.$$

Substituting from §§89, 90 we have

$$\tan (A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing each term of both numerator and denominator by $\cos A \cos B$, we have

$$\tan (A + B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \sin B}{\cos A \cos B}} = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \times \dfrac{\sin B}{\cos B}}.$$

But

$$\frac{\sin A}{\cos A} = \tan A \text{ and } \frac{\sin B}{\cos B} = \tan B.$$

Substituting, we have

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \qquad [26]$$

95. Formula for tan (*A* – *B*).—Substituting from §§91, 92 in

$$\tan (A - B) = \frac{\sin (A - B)}{\cos (A - B)}$$

and proceeding as above, we have

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \qquad [27]$$

96. Formulas for cot (*A* + *B*) and cot (*A* – *B*).—By a process similar to that used in §§94, 95 we find

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} \qquad [28]$$

and

$$\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A} \qquad [29]$$

97. Formulas for tan 2*A* and cot 2*A*.—Letting $B = A$ in the formulas for $\tan (A + B)$ and $\cot (A + B)$, (§§94, 96) we have

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad [30]$$

and

$$\cot 2A = \frac{\cot^2 A - 1}{2 \cot A} \qquad [31]$$

It should be noted that the formulas in §§93–97 have been proved under the same limitations on the angles A and B that were imposed in proving the corresponding formulas in §§89–92. When the formulas on sines and cosines are proved for all angles, the formulas on tangents and cotangents will follow for all angles by the argument used above. (See §133.)

EXERCISES

GROUP A

1. If $\sin A = 3/5$, find $\sin 2A$ and $\cos 2A$.

2. If $\cos A = \frac{1}{2}$, find $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$.

3. Using the formula for $\tan (45° + 30°)$, find $\tan 75°$.

Suggestion: For values of tan 45° and tan 30° see §11.

4. Using the formula for $\tan (45° - 30°)$, find $\tan 15°$.

5. Using the formula for $\cot (45° + 30°)$, find $\cot 75°$.

Suggestion: Note that $\cot A = 1/\tan A$. For values of tan 45° and tan 30° see §11.

6. Using the formula for $\cot (45° - 30°)$ find $\cot 15°$. Compare tan 75° and cot 15°, also cot 75° and tan 15°.

7. If $\tan A = \frac{1}{2}$ and $\tan B = 1/3$, find $\tan (A + B)$, also $\tan (A - B)$.

8. Prove that $\tan (A + 45°) \tan (A - 45°) = -1$.

Suggestion: Note that $\tan 45° = 1$.

9. Prove that $\tan (A + B) \tan (A - B) = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$.

10. Prove that $\dfrac{\cot (A + B)}{\cot (A - B)} = \dfrac{\sin A \cos A - \sin B \cos B}{\sin A \cos A + \sin B \cos B}$.

GROUP B

1. If $\cos A = 2/3$, find $\sin 2A$ and $\cos 2A$.

2. If $\sin A = 2/5$, find $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$.

3. Using the formula for $\tan (60° + 45°)$, find $\tan 105°$.

Suggestion: For values of tan 45° and tan 60°, see §11.

4. Using the formula for $\tan (60° + 60°)$ (or $\tan 2 \cdot 60°$) find $\tan 120°$. Compare this value with tan 60°.

5. Using the formula for $\cot (60° + 45°)$, find $\cot 105°$. Compare this in value with tan 105°.

6. Using the formula for $\cot (60° + 60°)$ (or $\cot 2 \cdot 60°$), find $\cot 120°$. Compare this value with cot 60°.

7. If $\tan A = 3$ and $\tan B = 1$, find $\tan (A + B)$, also $\tan (A - B)$.

8. Prove that $\cot (A + 45°) \cot (A - 45°) = -1$.

9. Prove that $\cot (A + B) \cot (A - B) = \dfrac{\cos^2 A - \sin^2 B}{\sin^2 A - \sin^2 B}$.

10. Prove that $\dfrac{\tan (A + B)}{\tan (A - B)} = \dfrac{\sin A \cos A + \sin B \cos B}{\sin A \cos A - \sin B \cos B}$.

98. The sum and difference of sines and cosines of two angles.—We have already developed the following four formulas:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Adding and subtracting the first two, we have,

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \qquad (1)$$
$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B. \qquad (2)$$

Similarly, from the last two we have,

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B \qquad (3)$$
$$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B. \qquad (4)$$

If we let $A + B = P$, and $A - B = Q$, then we have $A = \frac{P+Q}{2}$, $B = \frac{P-Q}{2}$.

Substituting these values in (1) – (4) we obtain

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2} \qquad [32]$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2} \qquad [33]$$

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2} \qquad [34]$$

$$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2} \qquad [35]$$

An especially important relation for the solution of oblique triangles is obtained from the first two of these formulas. Dividing [33] by [32]

$$\frac{\sin P - \sin Q}{\sin P + \sin Q} = \frac{\cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)}{\sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)}$$
$$= \tan \tfrac{1}{2}(P-Q) \cot \tfrac{1}{2}(P+Q)$$

or, since $\cot \frac{1}{2}(P+Q) = \frac{1}{\tan \frac{1}{2}(P+Q)}$,

$$\frac{\sin P - \sin Q}{\sin P + \sin Q} = \frac{\tan \frac{1}{2}(P-Q)}{\tan \frac{1}{2}(P+Q)}.$$

Note.—Formulas [32]—[35] make possible many useful reductions. Many of the exercises on the next page may be solved by means of these formulas.

99. Product of sines and cosines of two angles.—From the relations $\sin(A+B)+\sin(A-B)=2\sin A\cos B$, etc. (see page 69), we have

$$\sin A \cos B = \tfrac{1}{2}\sin(A+B)+\tfrac{1}{2}\sin(A-B) \tag*{[36]}$$
$$\cos A \sin B = \tfrac{1}{2}\sin(A+B)-\tfrac{1}{2}\sin(A-B) \tag*{[37]}$$
$$\cos A \cos B = \tfrac{1}{2}\cos(A+B)+\tfrac{1}{2}\cos(A-B) \tag*{[38]}$$
$$\sin A \sin B = -\tfrac{1}{2}\cos(A+B)+\tfrac{1}{2}\cos(A-B) \tag*{[39]}$$

These formulas enable us to express a product of sines and cosines as a sum.

EXERCISES

Prove:

1. $\sin 40° + \sin 20° = \cos 10°$.

Suggestion: $\sin 30° = 1/2$.

2. $\sin 60° + \sin 30° = \sqrt{2}\cos 15°$.

3. $\sin 50° - \sin 10° = \sqrt{3}\sin 20°$.

Suggestion: $\cos 30° = \frac{1}{2}\sqrt{3}$.

4. $\sin 70° - \sin 50° = \sin 10°$.

5. $\cos 25° + \cos 35° = \sqrt{3}\cos 5°$.

6. $\cos 75° + \cos 15° = \sqrt{2}\cos 30°$.

7. $\cos 85° - \cos 35° = -\sqrt{3}\sin 25°$.

8. $\cos 87° - \cos 27° = -\sin 57°$.

9. $\sin 6x + \sin 4x = 2\sin 5x\cos x$.

10. $\sin 6x - \sin 2x = 2\cos 4x\sin 2x$.

11. $\cos 5x + \cos 3x = 2\cos 4x\cos x$.

✓ **12.** $\cos 7x - \cos 3x = -2\sin 5x\sin 2x$.

13. $2\sin 3A\cos A = \sin 4A + \sin 2A$.

14. $2\cos 5A\sin 3A = \sin 8A - \sin 2A$.

15. $2\cos 4x\cos 2x = \cos 6x + \cos 2x$.

16. $2\sin 3x\sin 2x = -\cos 5x + \cos x$.

17. $2\sin(3x+2y)\cos(3x-2y) = \sin 6x + \sin 4y$.

18. $2\cos(3x+2y)\cos(3x-2y) = \cos 6x + \cos 4y$.

19. $\dfrac{\cos x + \cos y}{\cos x - \cos y} = -\cot\frac{1}{2}(x+y)\cot\frac{1}{2}(x-y)$.

Compare the formula for $\dfrac{\sin P - \sin Q}{\sin P + \sin Q}$ developed in §98.

20. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\dfrac{A+B}{2}$.

21. $\dfrac{\sin 3A + \sin A}{\cos 3A + \cos A} = \tan 2A$.

22. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \tan(A+B)\cot(A-B)$.

23. $\dfrac{\sin 3A + \sin A}{\sin 6A - \sin 2A} = \cos A\sec 4A$.

CHAPTER VIII

THE SOLUTION OF OBLIQUE TRIANGLES

We have now derived those trigonometric relations which are necessary for the development of formulas used in solving oblique triangles. As has already been noted, a triangle is determined if there are given:

(a) two angles and any side; (b) two sides and the included angle; (c) three sides. Since the third angle is known if two angles are given, it follows that problem (2) given on page 1 may be stated: "two angles and *any* side."

If two sides and the angle opposite one of them are given, it is sometimes possible to construct two different triangles fulfilling the conditions. Again, there may be only one such triangle, or possibly none. This case requires special discussion, and will be considered last.

It is evident, therefore, that the problem of solving oblique triangles falls under four essentially different cases:

Case I. Given two angles and any side.
Case II. Given two sides and the included angle.
Case III. Given three sides.
Case IV. Given two sides and the angle opposite one of them.

Inasmuch as computation in trigonometry is usually performed by the use of logarithms, formulas adapted to logarithmic computation will now be developed.

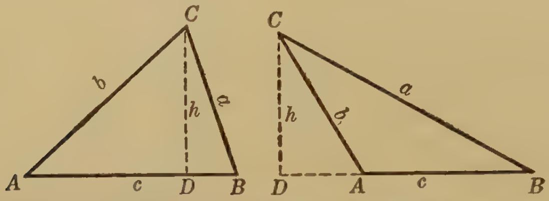
100. Arrangement of work.—One of the most effective aids to accuracy in computation, especially in logarithmic computation, is careful arrangement of work. In the following pages sample solutions of problems are given which illustrate convenient arrangements. A skeleton outline of the whole solution should be made before looking up any logarithms.

101. Checking.—It is *absolutely* necessary that the computer should *know* that his results are correct. For this reason *all* work must be checked. Methods of checking are given under each case.

It is said that a computing engineer uses two-thirds of his time in direct computing, and one-third of it in checking. In practice there is no answer book, but the results *must* be accurate.

102. The law of sines.—*The sides of a triangle are proportional to the sines of the opposite angles.*

In symbols, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad [40]$$



Proof: In the triangle ABC (either figure), drop the perpendicular h from the vertex C to the side AB (extended if necessary).

Then, $\sin A = \frac{h}{b}$ and $\sin B = \frac{h}{a}$.

Dividing, $$\frac{\sin A}{\sin B} = \frac{h/b}{h/a} = \frac{a}{b}.$$

Hence, $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly we may prove

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{and} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

103. Solution of triangles: case I.—When two angles and a side of a triangle are given, the law of sines is used in solving the triangle.

Suppose the given parts are A, B, and side b. The first step is to find C by means of the formula $C = 180° - (A + B)$.

From $\frac{a}{\sin A} = \frac{b}{\sin B}$ we obtain $a = b\,\frac{\sin A}{\sin B}$.

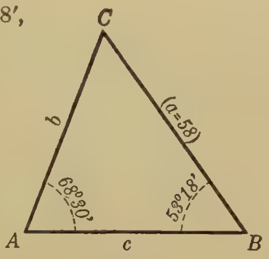
Similarly, from $\frac{c}{\sin C} = \frac{b}{\sin B}$ we obtain $c = b\,\frac{\sin C}{\sin B}$.

Check: To check these values of a and c, use the formula

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}.$$

If this relation holds, the values are *very* likely correct. If the relation does not hold, the values cannot be correct. These quotients should be found by long division, for if they are found by using the same logarithms that were used in solving the problem, an error in finding a logarithm or an anti-logarithm may not be disclosed. For another check see page 167.

104. Computation.—The actual work of computing in case I is shown in the following:



Example. Given, $\angle A = 68^\circ\ 30'$, $\angle B = 53^\circ\ 18'$, $a = 58$. Find $\angle C$, and sides b and c.
First, $\angle C = 180^\circ - (\angle A + \angle B) = 58^\circ\ 12'$.

From $\dfrac{b}{a} = \dfrac{\sin B}{\sin A}$ and $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$ we have

$$b = a\,\frac{\sin B}{\sin A},\quad c = a\,\frac{\sin C}{\sin A}.$$

Hence $\log b = \log a + \log \sin B - \log \sin A$,
and $\log c = \log a + \log \sin C - \log \sin A$.
Arrange all the work as follows before beginning to use the tables.

$$\begin{array}{rlcrl}
\log a = & & \qquad & \log a = & \\
\log \sin B = & \underline{\qquad\quad} & & \log \sin C = & \underline{\qquad\quad} \\
\log \sin A = & & & \log \sin A = & \\
\log b = & & & \log c = & \\
b = & & & c = &
\end{array}$$

This skeleton should then be filled in as follows:

$$\begin{array}{rlcrl}
\log a = & 1.76343 & \qquad & \log a = & 1.76343 \\
\log \sin B = & \underline{9.90405} & & \log \sin C = & \underline{9.92936} \\
 & 11.66748 & & & 11.69279 \\
\log \sin A = & \underline{9.96868} & & \log \sin A = & \underline{9.96868} \\
\log b = & 1.69880 & & \log c = & 1.72411 \\
b = & 49.980 & & c = & 52.980
\end{array}$$

Check: Use $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, or better $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

From the table of natural functions, $\sin B = .80178$, $\sin C = .84989$

$$\begin{array}{rlcrl}
 & \underline{\quad .01604+} & \qquad & & \underline{\quad .01604+} \\
49.98) & 80178 & & 52.98) & 84989 \\
 & \underline{4998} & & & \underline{5298} \\
 & 30198 & & & 32009 \\
 & \underline{29988} & & & \underline{31788} \\
 & \quad 21000 & & & \quad 22100 \\
 & \quad \underline{19992} & & & \quad \underline{21192}
\end{array}$$

EXERCISES

1. Using the figure of §102, draw the altitude on the side AC and prove that $a/\sin A = c/\sin C$.

Find the elements not given in the following:

2. $\begin{cases} \angle A = 48^\circ\ 37' \\ \angle B = 73^\circ\ 12' \\ a = 181.8 \end{cases}$ **3.** $\begin{cases} \angle A = 61^\circ\ 28' \\ \angle B = 24^\circ\ 19' \\ b = 381.6 \end{cases}$ **4.** $\begin{cases} \angle A = 84^\circ\ \ 7' \\ \angle B = 23^\circ\ 58' \\ c = 94.36 \end{cases}$

105. The law of tangents.—By the law of sines,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Taking this proportion by composition and division,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.[1]$$

But we have already proved (§98) that

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

Hence,

$$\boldsymbol{\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}}. \qquad [41]$$

Similarly,

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} \text{ and } \frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(C-A)}{\tan \frac{1}{2}(C+A)}.$$

These formulas are referred to as the *law of tangents.*

106. Solution of triangles: case II.—If we have given two sides and the included angle, the law of tangents formula may be used in the solution of the triangle.

If the parts given are a, b, and C, then

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

The quantity $A + B$ may be found by subtracting C from 180°. Then the only unknown in this equation is $\tan \frac{1}{2}(A - B)$. Solving the formula for this quantity, we have

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B).$$

When $\frac{1}{2}(A - B)$ has been found, A and B are obtained from the relations:

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

The remaining side c may then be found by using the law of sines as in case I. (Use $c = a \sin C/\sin A$.)

Check: To check see whether the relation $c = \dfrac{b \sin C}{\sin B}$ is satisfied.

---

[1] Given $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a-b}{b} = \frac{c-d}{d}$. Hence it follows by division that $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

107. Computation.—If in a triangle ABC, angle A and sides b and c are given, the first step in the solution is to find $b + c$, $b - c$, and $\frac{1}{2}(B + C)$

From $\frac{1}{2}A + \frac{1}{2}(B + C) = 90°$ we have

$$\tfrac{1}{2}(B + C) = 90° - \tfrac{1}{2}A.$$

Example. Given $\angle A = 83° 44'$, $b = 78.4$, and $c = 61.3$, find angles B and C and side a.

Skeleton solution: We first find $\frac{1}{2}(B + C) = 90° - 41° 52' = 48° 8'$, $b + c = 139.7$, and $b - c = 17.1$.

Since $\tan \frac{1}{2}(B - C) = \frac{b - c}{b + c} \tan \frac{1}{2}(B + C)$, and therefore $\log \tan \frac{1}{2}(B - C) = \log (b - c) + \log \tan \frac{1}{2}(B + C) - \log (b + c)$, we arrange the following skeleton solution:

$\log (b - c) =$
$\log \tan \frac{1}{2}(B + C) =$
$\operatorname{colog} (b + c)$[1] $=$ ______
$\log \tan \frac{1}{2}(B - C) =$
$\frac{1}{2}(B - C) =$
$\frac{1}{2}(B + C) =$
$B =$
$C =$

The side a is found from the formula

$$a = \frac{b \sin A}{\sin B}$$

and therefore $\log a = \log b + \log \sin A - \log \sin B$

(Skeleton solution)

$\log b =$
$\log \sin A =$
$\operatorname{colog} \sin B =$
$\log a =$
$a =$

Notice that B is found by adding $\frac{1}{2}(B - C)$ and $\frac{1}{2}(B + C)$ and C is found by subtracting $\frac{1}{2}(B - C)$ from $\frac{1}{2}(B + C)$.

Check: Verify that $c = \frac{a \sin C}{\sin A}$

It is not a check even on the angles to show that $A + B + C = 180°$. This would show only that B and C satisfy the equation

$$\tfrac{1}{2}(B + C) = 90° - \tfrac{1}{2}A.$$

This equation may be satisfied and still both B and C may be wrong.

Note.—If in an example c is given greater than b, we use the forms

$$\tan \tfrac{1}{2}(C - B) = \frac{c - b}{c + b} \tan \tfrac{1}{2}(C + B).$$

This avoids the use of negative factors and negative angles.

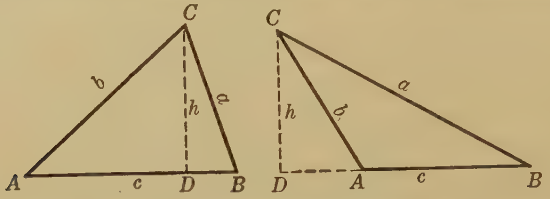
EXERCISES

Solve the following triangles:

1. $\begin{cases} \angle A = 67° 28' \\ b = 237.8 \\ c = 201.6 \end{cases}$ **2.** $\begin{cases} \angle B = 43° 57' \\ a = 91.7 \\ c = 103.4 \end{cases}$ **3.** $\begin{cases} \angle C = 85° 37' \\ a = 147.6 \\ b = 89.3 \end{cases}$

[1] Expressions containing only one factor in the numerator and one in the denominator should be evaluated by subtracting logarithms. If there are two or more factors in either, cologs may be used.

108. The law of cosines.—*In any triangle, the square of one side is equal to the sum of the squares of the other two sides diminished by twice their product multiplied by the cosine of the included angle.*



In each figure drop a perpendicular from the vertex C to the side AB.

In either figure, $a^2 = h^2 + \overline{BD}^2$,

while $\overline{BD}^2 = (c - AD)^2 = c^2 - 2c \times AD + \overline{AD}^2$.

(Note that in the figure to the right AD is negative.)

Hence $a^2 = h^2 + \overline{AD}^2 + c^2 - 2c \times AD$.

But, $h^2 + \overline{AD}^2 = b^2$ and $AD = b \cos A$.

Making these substitutions we have

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad [42]$$

Similarly, it may be proved that

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Hence we have proved the *law of cosines* stated above.

The law of cosines is developed here only for its application in deriving formulas for finding the angles of a triangle in terms of its sides. This formula itself is not adapted to logarithmic work. It may, however, be used with the natural functions for the solution of problems.

Example. From the law of cosines, determine the value of the cosine of any angle of a triangle in terms of the three sides of the triangle.

Solution: Solving for $\cos A$ the equation

$$a^2 = b^2 + c^2 - 2\,bc \cos A,$$

we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

EXERCISES

1. Find the angles of a triangle whose sides are 4, 5, 6.

Suggestion: Find the angles by means of the law of cosines. Use table of natural functions.

2. Find A and B in the triangle ABC, if $a = 8$, $b = 10$ and $c = 9$.

3. Find A if $a = 14.3$, $b = 17.2$, $c = 19.7$.

109. The half-angle formulas.—By formulas [24] and [25] §93 we have

$$\sin^2 \tfrac{1}{2} A = \frac{1 - \cos A}{2} \qquad \cos^2 \tfrac{1}{2} A = \frac{1 + \cos A}{2}$$

By means of these formulas and the law of cosines we are enabled to express the functions of the half-angles in terms of the sides of the triangle.

Let s equal one-half the perimeter of the triangle; that is, let $a + b + c = 2s$.

Subtracting successively $2a$, $2b$, $2c$, from both sides of this equality, we obtain

$$\begin{aligned} b + c - a &= 2s - 2a = 2(s - a) \\ a + c - b &= 2s - 2b = 2(s - b) \\ a + b - c &= 2s - 2c = 2(s - c). \end{aligned}$$

By the law of cosines, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Hence, $1 - \cos A = 1 - \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{2bc - b^2 - c^2 + a^2}{2bc}$

$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}$$

$$= \frac{2(s - c)(s - b)}{bc}.$$

Similarly, $1 + \cos A = \dfrac{2s(s - a)}{bc}$.

Therefore, $\sin^2 \tfrac{1}{2} A = \dfrac{(s - b)(s - c)}{bc}$ and $\cos^2 \tfrac{1}{2} A = \dfrac{s(s - a)}{bc}$.

Hence, $$\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \qquad [43]$$

and $$\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}} \qquad [44]$$

Then by division, $$\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} \qquad [45]$$

NOTE. Since s is expressed in terms of a, b, c, it follows that the above formulas give $\sin \tfrac{1}{2} A$, $\cos \tfrac{1}{2} A$, and $\tan \tfrac{1}{2} A$ in terms of the sides of the triangle.

Sight Work

1. Express $\sin \tfrac{1}{2} B$ and $\sin \tfrac{1}{2} C$ in terms of the sides a, b, c, of the $\triangle ABC$.
2. Express $\cos \tfrac{1}{2} B$ and $\cos \tfrac{1}{2} C$ in terms of the sides a, b, c of the $\triangle ABC$.
3. Express $\tan \tfrac{1}{2} B$ and $\tan \tfrac{1}{2} C$ in terms of the sides a, b, c of the $\triangle ABC$.

110. Solution of triangles: case III.—In solving a triangle in which the three sides are given any one of the formulas, [43], [44], [45] may be used.

However, the formulas given below are the most convenient, since they require the smallest number of different logarithms for finding all the angles.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

$$\text{or,}\quad \tan\frac{A}{2} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad [46]$$

$$\text{Similarly,}\quad \tan\frac{B}{2} = \frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\text{and}\quad \tan\frac{C}{2} = \frac{1}{s-c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

These differ only in the factors before the radical sign. The logarithm of the quantity under the radical sign may therefore be found once for all and used for all the angles.

It is usual to write

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\text{Then,}\quad \tan\frac{A}{2} = \frac{r}{s-a},\ \tan\frac{B}{2} = \frac{r}{s-b},\ \tan\frac{C}{2} = \frac{r}{s-c}$$

$\log r = \frac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) - \log s]$ and $\log\tan\frac{1}{2}A = \log r - \log(s-a)$, etc.

We may arrange the following skeleton solution:

$a =$
$b =$
$c =$
$2s =$
$s =$
$s - a =$
$s - b =$
$s - c =$
$\log(s-a) =$
$\log(s-b) =$
$\log(s-c) =$
$\operatorname{colog} s =$
$2)$
$\log r =$

$\log r =$
$\log(s-a) =$
$\log\tan\frac{1}{2}A =$
$\frac{1}{2}A =$
$\log r =$
$\log(s-b) =$
$\log\tan\frac{1}{2}B =$
$\frac{1}{2}B =$
$\log r =$
$\log(s-c) =$
$\log\tan\frac{1}{2}C =$
$\frac{1}{2}C =$

EXERCISES IN THE SOLUTION OF TRIANGLES

GROUP A

$$\mathbf{1.}\begin{cases}\angle A = 73^\circ\ 43' \\ \angle B = 64^\circ\ 21' \\ a = 191.8\end{cases}$$

$$\mathbf{2.}\begin{cases}\angle A = 47^\circ\ 56' \\ \angle C = 67^\circ\ 43' \\ c = 21.92\end{cases}$$

$$\mathbf{3.}\begin{cases}\angle B = 16^\circ\ 42' \\ \angle C = 78^\circ\ 28' \\ b = 297.4\end{cases}$$

$$\mathbf{4.}\begin{cases}\angle B = 61^\circ\ 29' \\ \angle C = 43^\circ\ 46' \\ c = 162.9\end{cases}$$

$$\mathbf{5.}\begin{cases}\angle A = 73^\circ\ 24' \\ b = 191.6 \\ c = 103.4\end{cases}$$

$$\mathbf{6.}\begin{cases}\angle B = 49^\circ\ 37' \\ a = 27.6 \\ c = 34.3\end{cases}$$

$$\mathbf{7.}\begin{cases}\angle C = 53^\circ\ 26' \\ a = 26.74 \\ b = 37.5\end{cases}$$

$$\mathbf{8.}\begin{cases}\angle A = 32^\circ\ 47' \\ \angle B = 67^\circ\ 32' \\ c = 214.7\end{cases}$$

$$\mathbf{9.}\begin{cases}a = 64.79 \\ b = 58.64 \\ c = 47.21\end{cases}$$

$$\mathbf{10.}\begin{cases}a = 37.41 \\ b = 28.96 \\ c = 64.7\end{cases}$$

$$\mathbf{11.}\begin{cases}a = 54.5 \\ b = 68.7 \\ c = 92.3\end{cases}$$

$$\mathbf{12.}\begin{cases}a = 71.7 \\ b = 84.5 \\ c = 107.6\end{cases}$$

GROUP B

$$\mathbf{1.}\begin{cases}\angle A = 37^\circ\ 14' \\ \angle B = 74^\circ\ 29' \\ b = 124.7\end{cases}$$

$$\mathbf{2.}\begin{cases}\angle A = 69^\circ\ 13' \\ \angle B = 29^\circ\ 48' \\ a = 37.6\end{cases}$$

$$\mathbf{3.}\begin{cases}\angle B = 52^\circ\ 27' \\ \angle C = 63^\circ\ 25' \\ a = 147.6\end{cases}$$

$$\mathbf{4.}\begin{cases}\angle B = 81^\circ\ 34' \\ \angle C = 26^\circ\ 25' \\ c = 214.6\end{cases}$$

$$\mathbf{5.}\begin{cases}\angle B = 47^\circ\ 26' \\ a = 183.6 \\ c = 126.7\end{cases}$$

$$\mathbf{6.}\begin{cases}\angle A = 69^\circ\ 31' \\ b = 54.6 \\ c = 42.7\end{cases}$$

$$\mathbf{7.}\begin{cases}\angle B = 65^\circ\ 45' \\ a = 98.7 \\ c = 74.6\end{cases}$$

$$\mathbf{8.}\begin{cases}\angle C = 71^\circ\ 47' \\ a = 138.6 \\ b = 293.4\end{cases}$$

$$\mathbf{9.}\begin{cases}a = 12.8 \\ b = 13.9 \\ c = 17.8\end{cases}$$

$$\mathbf{10.}\begin{cases}a = 39.6 \\ b = 47.8 \\ c = 31.2\end{cases}$$

$$\mathbf{11.}\begin{cases}a = 126.7 \\ b = 139.8 \\ c = 187.3\end{cases}$$

$$\mathbf{12.}\begin{cases}a = 461.7 \\ b = 617.4 \\ c = 562.4\end{cases}$$

GROUP C

$$\mathbf{1.}\begin{cases}\angle A = 39^\circ\ 14' \\ \angle B = 67^\circ\ 18' \\ a = 291.7\end{cases}$$

$$\mathbf{2.}\begin{cases}\angle B = 72^\circ\ 31' \\ a = 161.7 \\ c = 214.5\end{cases}$$

$$\mathbf{3.}\begin{cases}a = 41 \\ b = 62 \\ c = 58\end{cases}$$

$$\mathbf{4.}\begin{cases}\angle A = 54^\circ\ 16' \\ \angle B = 63^\circ\ 18' \\ b = 245.6\end{cases}$$

$$\mathbf{5.}\begin{cases}\angle A = 84^\circ\ 36' \\ b = 36.7 \\ c = 56.8\end{cases}$$

$$\mathbf{6.}\begin{cases}a = 317 \\ b = 486 \\ c = 397\end{cases}$$

$$\mathbf{7.}\begin{cases}\angle B = 69^\circ\ 12' \\ \angle C = 49^\circ\ 52' \\ c = 84.7\end{cases}$$

$$\mathbf{8.}\begin{cases}\angle B = 52^\circ\ 39' \\ a = 24.7 \\ c = 28.6\end{cases}$$

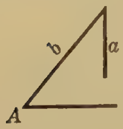
$$\mathbf{9.}\begin{cases}a = 1082 \\ b = 2176 \\ c = 1455\end{cases}$$

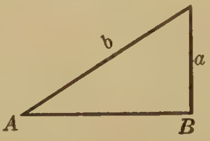
$$\mathbf{10.}\begin{cases}\angle B = 43^\circ\ 46' \\ C = 67^\circ\ 21' \\ a = 189.7\end{cases}$$

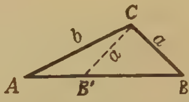
$$\mathbf{11.}\begin{cases}\angle C = 55^\circ\ 25' \\ a = 187.6 \\ b = 214.7\end{cases}$$

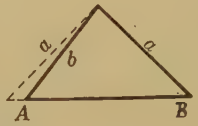
$$\mathbf{12.}\begin{cases}a = 1004 \\ b = 1276 \\ c = 1384\end{cases}$$

111. Case IV (the ambiguous case).—If it is required to find a triangle having two given sides a and b and a given angle, as A, opposite one of them, then a number of possibilities arise, as indicated by the figures:









(1) If $a < b \sin A$, then there is no solution of the problem.

(2) If $a = b \sin A$, then there is only one solution, and the triangle is right angled at B.

(3) If $a < b$ and also $a > b \sin A$, then there are two triangles possessing the given properties.

That is, there are two solutions of the problem. In this case the angles opposite the side b ($\angle ABC$ and $\angle AB'C$) are supplementary. Hence, angle B or angle B' of the figure must be obtuse.

(4) If $a = b$ or $a > b$, there is only one solution, since in that case the angle opposite b is equal to or less than the angle opposite a. That is, angle B cannot be obtuse.

112. Computation.—The various possibilities just stated will appear when the attempt is made to compute the triangle. This is shown in the following examples:

Example 1. Compute the triangle if $a = 28$, $b = 46$, and $\angle A = 71° 43'$.

Solution: From $\frac{a}{\sin A} = \frac{b}{\sin B}$, we find

$$\sin B = \frac{b \sin A}{a},$$

and $\log \sin B = \log b + \log \sin A - \log a$.

$$\begin{aligned} \text{Then:} \quad \log b &= 1.66276 \\ \log \sin A &= \underline{9.97750} \\ &\ 1.64026 \\ \log a &= 1.44716 \\ \log \sin B &= 10.19310 - 10 \end{aligned}$$

But $\log \sin B = 0.19310$ is at once seen to be impossible, for this would make $\sin B$ greater than 1. Hence in this case there is no solution.

Example 2. Compute the triangle if $a = 12$, $b = 24$, and $\angle A = 30°$.

Solution: Use $\sin B = \frac{b \sin A}{a}$

$$\begin{aligned} \log b &= 1.38021 \\ \log \sin A &= \underline{9.69897} \\ &\ 11.07918 \\ \log a &= \underline{1.07918} \\ \log \sin B &= 0.00000 \end{aligned}$$

Since $\log \sin B = 0$, we know that $\sin B = 1$, and hence $B = 90$. That is, the triangle ABC is right angled at B. In this case $a = b \sin A$.

Example 3. Compute the triangle if $a = 37.8$, $b = 43.2$ and $\angle A = 57^\circ 45'$.

Solution: Use formula $\sin B = \dfrac{b \sin A}{a}$.

Then $\log \sin B = \log b + \log \sin A - \log a$.

$$\begin{array}{rl} \log b &= 1.63548 \\ \log \sin A &= \underline{9.92723} \\ & \ 11.56271 \\ \log a &= \underline{1.57749} \\ \log \sin B &= 9.98522 \end{array}$$

However, since $\sin x = \sin (180^\circ - x)$ the angle B may also be $180^\circ - 75^\circ 8' = 104^\circ 52'$.

$B = 75^\circ 8' 15''$, or $75^\circ 8'$ to the nearest minute.

Then,
$$C = 180^\circ - (57^\circ 45' + 75^\circ 8') = 47^\circ 7' \text{ or}$$
$$C = 180^\circ - (57^\circ 45' + 104^\circ 52') = 17^\circ 23'.$$

Using the sine formula, we now find the two possible values of the side c by using in succession these two values of angle C.

113. Ambiguity of case IV as shown in the computation.—In all the examples given in §112 the formula $\sin B = \dfrac{b \sin A}{a}$ is used for the purpose of finding the angle B. From this formula the various possible cases are at once apparent.

(1) If $\sin B > 1$, i.e., $a < b \sin A$, the problem is obviously impossible, since a sine cannot be greater than unity.

(2) If $\sin B = 1$, i.e., $a = b \sin A$, then the only possible angle is a right angle, since $\sin 90^\circ = 1$.

(3) If $\sin B < 1$, i.e., $a > b \sin A$, then there are two angles, each less than 180°, which have the required sine.

EXERCISES

In the following give all possible solutions.

1. $\begin{cases} \angle A = 38^\circ 3' \\ a = 127.3 \\ b = 139.6 \end{cases}$

2. $\begin{cases} \angle A = 30^\circ \\ a = 137.4 \\ b = 274.8 \end{cases}$

3. $\begin{cases} \angle A = 62^\circ 30' \\ a = 26.7 \\ b = 49.3 \end{cases}$

4. $\begin{cases} \angle B = 67^\circ 18' \\ a = 59.3 \\ b = 67.8 \end{cases}$

5. $\begin{cases} \angle B = 47^\circ 52' \\ c = 136.4 \\ b = 123.2 \end{cases}$

6. $\begin{cases} \angle B = 17^\circ 13' \\ a = 8.7 \\ b = 36.2 \end{cases}$

7. $\begin{cases} \angle C = 31^\circ 24' \\ c = 49.8 \\ b = 41.6 \end{cases}$

8. $\begin{cases} \angle C = 72^\circ 41' \\ a = 29.8 \\ c = 22.4 \end{cases}$

9. $\begin{cases} \angle C = 59^\circ 37' \\ c = 294.6 \\ a = 217.2 \end{cases}$

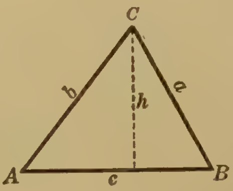
10. $\begin{cases} A = 47^\circ 39' \\ a = 57.8 \\ b = 127.1 \end{cases}$

11. $\begin{cases} A = 61^\circ 24' \\ b = 81.6 \\ a = 73.4 \end{cases}$

12. $\begin{cases} B = 33^\circ 25' \\ c = 216 \\ b = 127 \end{cases}$

114. Area of a triangle.—The area of a triangle may be found in terms of any set of elements which are sufficient to determine the triangle. We therefore have three cases to consider:

Case I. Given two sides and the included angle.
Case II. Given two angles and any side.
Case III. Given three sides.



Case I. Let the given parts be b, c, A.

In the triangle ABC, let h be the altitude upon the side c.

Denoting the area by S we have

$$S = \tfrac{1}{2}\, ch$$

But $h = b \sin A$

Hence, $$S = \tfrac{1}{2}\, bc \sin A \qquad [47]$$

Similarly, $S = \tfrac{1}{2}\, ac \sin B = \tfrac{1}{2}\, ab \sin C$.

Case II. Let the given parts be a, B, C. Then A can be found at once. From the law of sines,

$$c = \frac{a \sin C}{\sin A} \quad \text{and from Case I,} \quad S = \tfrac{1}{2}\, ac \sin B.$$

Hence, $$S = \frac{a^2 \sin B \sin C}{2 \sin A} \qquad [48]$$

Case III. Let the given parts be a, b, c.

Then $\sin \tfrac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\cos \tfrac{1}{2} A = \sqrt{\dfrac{s(s-a)}{bc}}$.

Since $\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$ (See § 93),

it follows that $$\sin A = 2 \cdot \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$
$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Substituting in the formula of Case I, we have

$$S = \sqrt{s(s-a)(s-b)(s-c)} \qquad [49]$$

which is known as Heron's formula.

EXERCISES

Find the areas of the following triangles.

1. $\begin{cases} a = 134.5 \\ b = 241.7 \\ \angle C = 54° 47' \end{cases}$

2. $\begin{cases} \angle A = 73° 42' \\ b = 37.5 \\ c = 47.8 \end{cases}$

3. $\begin{cases} \angle A = 41° 56' \\ \angle B = 67° 49' \\ c = 210.6 \end{cases}$

4. $\begin{cases} \angle A = 57° 32' \\ \angle B = 81° 27' \\ a = 397.4 \end{cases}$

5. $\begin{cases} a = 62.7 \\ b = 78.4 \\ c = 93.6 \end{cases}$

6. $\begin{cases} a = 386.7 \\ b = 296.4 \\ c = 487.6 \end{cases}$

115. Radius of an inscribed circle.—If r is the radius of the circle inscribed in a triangle whose sides are a, b, c, then the area S of the triangle is

$$S = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \tfrac{1}{2}r(a+b+c) = \tfrac{1}{2}r\cdot 2s = rs.$$

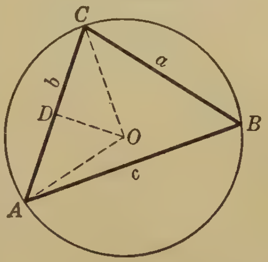
But by [49], $S = \sqrt{s(s-a)(s-b)(s-c)} = rs.$

Hence, $$r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)} \qquad [50]$$

116. Radius of a circumscribed circle.—In the figure let O be the center of the circumscribed circle, with radius R, and let OD be the perpendicular bisector of the side AC. Then $\angle AOC$ is measured by the arc AC, while $\angle ABC$ or $\angle B$ is measured by half of this arc. Hence,

$$\angle AOC = 2\angle B \text{ and } \angle AOD = \angle B.$$

But $AO = R$ and $AD = b/2$.



Then, $R \sin AOD = R \sin B = b/2$, and $R = \dfrac{b}{2\sin B}.$

Since $S = \tfrac{1}{2}ac \sin B$, we have $\sin B = 2S/ac$.

Hence, $R = \dfrac{b}{2\dfrac{2S}{ac}} = \dfrac{abc}{4S},$

and we have, $$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad [51]$$

EXERCISES

Find the radii of the circles inscribed in the following triangles.

$1. \begin{cases} a = 12 \\ b = 14 \\ c = 16 \end{cases}$ $2. \begin{cases} a = 25.6 \\ b = 30.4 \\ c = 35.8 \end{cases}$ $3. \begin{cases} a = 40 \\ b = 42 \\ c = 44 \end{cases}$ $4. \begin{cases} a = 55 \\ b = 60 \\ c = 65 \end{cases}$

$5. \begin{cases} \angle A = 71° 50' \\ b = 28 \\ c = 30 \end{cases}$ In Exercise 5 find the unknown side before finding the required radius.

Find the radii of the circles circumscribed about the following triangles.

$6. \begin{cases} a = 47 \\ b = 67 \\ c = 70 \end{cases}$ $7. \begin{cases} A = 58° \ 7' \\ B = 47° 15' \\ c = 27.8 \end{cases}$ $8. \begin{cases} \angle A = 67° 24' \\ a = 92.6 \\ \angle B = 43° 21' \end{cases}$ $9. \begin{cases} \angle B = 37° 17' \\ a = 189.7 \\ c = 431.2 \end{cases}$

EXERCISES IN THE SOLUTION OF TRIANGLES

Group A

1. $\left\{\begin{array}{l}\angle A = 61^\circ\ 23' \\ \angle B = 46^\circ\ 18' \\ a = 261.3\end{array}\right.$

2. $\left\{\begin{array}{l}\angle A = 82^\circ\ 12' \\ b = 12.32 \\ c = 17.66\end{array}\right.$

3. $\left\{\begin{array}{l}a = 1614 \\ b = 1123 \\ c = 917\end{array}\right.$

4. $\left\{\begin{array}{l}\angle A = 54^\circ\ 20' \\ a = 96.5 \\ b = 118.3\end{array}\right.$

5. $\left\{\begin{array}{l}a = 3.525 \\ c = 2.165 \\ \angle B = 108^\circ\ 25'\end{array}\right.$

6. $\left\{\begin{array}{l}\angle B = 34^\circ\ 56' \\ \angle C = 61^\circ\ 12' \\ c = 1463\end{array}\right.$

7. $\left\{\begin{array}{l}\angle C = 76^\circ\ 38' \\ a = 116.5 \\ c = 161.4\end{array}\right.$

8. $\left\{\begin{array}{l}\angle C = 29^\circ\ 5' \\ a = 42.27 \\ b = 51.45\end{array}\right.$

9. $\left\{\begin{array}{l}a = 46.25 \\ b = 37.36 \\ c = 51.63\end{array}\right.$

10. $\left\{\begin{array}{l}\angle B = 37^\circ\ 31' \\ b = 400.4 \\ c = 763.5\end{array}\right.$

11. $\left\{\begin{array}{l}\angle A = 27^\circ\ 42' \\ \angle B = 39^\circ\ 28' \\ c = 1.163\end{array}\right.$

12. $\left\{\begin{array}{l}a = 0.1826 \\ b = 0.2716 \\ c = 0.2065\end{array}\right.$

Group B

1. $\left\{\begin{array}{l}a = 271.6 \\ b = 185.8 \\ c = 223.6\end{array}\right.$

2. $\left\{\begin{array}{l}\angle B = 86^\circ\ 56' \\ a = 265.5 \\ c = 826.3\end{array}\right.$

3. $\left\{\begin{array}{l}\angle A = 42^\circ\ 45' \\ \angle B = 67^\circ\ 35' \\ a = 76.32\end{array}\right.$

4. $\left\{\begin{array}{l}\angle B = 39^\circ\ 18' \\ b = 3656 \\ c = 5260\end{array}\right.$

5. $\left\{\begin{array}{l}\angle B = 108^\circ\ 25' \\ \angle C = 30^\circ\ 40' \\ c = 25.63\end{array}\right.$

6. $\left\{\begin{array}{l}\angle A = 27^\circ\ 50' \\ b = 756.5 \\ c = 265.5\end{array}\right.$

7. $\left\{\begin{array}{l}\angle A = 38^\circ\ 28' \\ \angle B = 59^\circ\ 12' \\ c = 165.0\end{array}\right.$

8. $\left\{\begin{array}{l}\angle C = 23^\circ\ 42' \\ a = 4127 \\ b = 6133\end{array}\right.$

9. $\left\{\begin{array}{l}\angle C = 13^\circ\ 54' \\ a = 0.0134 \\ c = 0.0177\end{array}\right.$

10. $\left\{\begin{array}{l}a = 267.8 \\ b = 183.3 \\ c = 401.5\end{array}\right.$

11. $\left\{\begin{array}{l}\angle A = 67^\circ\ 35' \\ a = 78.52 \\ b = 82.75\end{array}\right.$

12. $\left\{\begin{array}{l}a = 65.25 \\ b = 46.83 \\ c = 51.62\end{array}\right.$

Group C

1. $\left\{\begin{array}{l}\angle A = 70^\circ\ 37' \\ a = 21.34 \\ c = 17.58\end{array}\right.$

2. $\left\{\begin{array}{l}\angle A = 29^\circ\ 19' \\ \angle B = 60^\circ\ 41' \\ c = 421.8\end{array}\right.$

3. $\left\{\begin{array}{l}a = 255.5 \\ b = 283.7 \\ c = 166.0\end{array}\right.$

4. $\left\{\begin{array}{l}\angle B = 66^\circ\ 13' \\ \angle C = 48^\circ\ 12' \\ b = 2565\end{array}\right.$

5. $\left\{\begin{array}{l}\angle A = 117^\circ\ 30' \\ b = 1.107 \\ c = 1.623\end{array}\right.$

6. $\left\{\begin{array}{l}\angle C = 112^\circ\ 21' \\ a = 315.6 \\ c = 382.5\end{array}\right.$

7. $\left\{\begin{array}{l}\angle B = 12^\circ\ 48' \\ a = 261.6 \\ c = 628.2\end{array}\right.$

8. $\left\{\begin{array}{l}a = 12.61 \\ b = 5.83 \\ c = 14.26\end{array}\right.$

9. $\left\{\begin{array}{l}\angle C = 58^\circ\ 28' \\ a = 175.6 \\ b = 181.4\end{array}\right.$

10. $\left\{\begin{array}{l}a = 6.387 \\ b = 8.566 \\ c = 9.053\end{array}\right.$

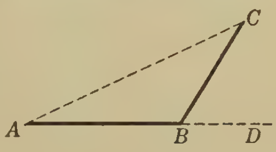
11. $\left\{\begin{array}{l}\angle A = 38^\circ\ 38' \\ a = 201.6 \\ b = 322.9\end{array}\right.$

12. $\left\{\begin{array}{l}\angle B = 26^\circ\ 32' \\ \angle C = 83^\circ\ 14' \\ c = 210.3\end{array}\right.$

EXERCISES

Group A

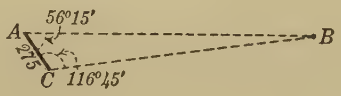
1. To find the angle at which the side of a railway embankment is inclined to the horizontal, a distance $BC = 28$ feet is measured up the embankment, a distance $BA = 42$ feet on the level ground and directly away from it, and then a surveyor's chain is stretched from A to C, finding this distance to be 63 feet. Using these data, find the angle DBC in the figure.



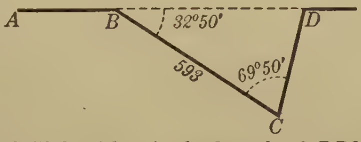
Suggestion: First find $\angle ABC$.

Question. Could the angle ABC be measured easily by means of a surveyor's transit?

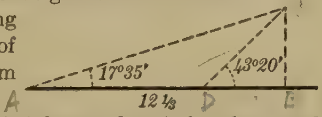
2. To find the distance from the point A on one side of a river to the point B on the other side, a line $AC = 275$ feet is measured and the angles A and C are found to be 56° 15′ and 116° 45′ respectively. Compute AB.



3. In surveying a straight line an obstacle is encountered, so it becomes necessary to make a detour from B to C (see figure), returning to the line at D. If $\angle ABC = 147°\ 10'$, $\angle C = 69°\ 50'$, and $BC = 593$ feet, what must be the length of CD? What is the length of BD?



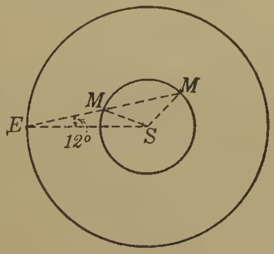
4. From a ship sailing in a straight course a lighthouse is observed to be 17° 35′ to the left of the course. After sailing $12\frac{1}{3}$ miles the lighthouse is 43° 20′ to the left of the course. What is the shortest distance from the lighthouse to the course of the ship?



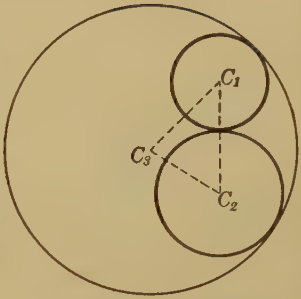
5. The diagonals of a parallelogram are 17.4 feet and 21.7 feet long and meet in an angle of 49° 10′. Find the sides of the parallelogram and also its area.

6. Two warships are observed to fire simultaneously, the reports reaching the observer in 6 seconds and 11 seconds, respectively. How far apart are the ships if they subtend an angle of 59° 40′ from the point of the observer and if sound is traveling at the rate of 1100 feet per second?

7. The distances in miles from the earth and Mercury to the sun are about 93,000,000 and 36,000,000, respectively. How far is Mercury from us when she appears at a distance of 12° from the sun? Note that two answers are possible. Give both of them.

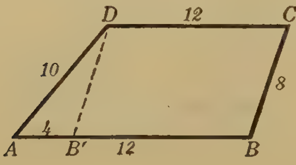


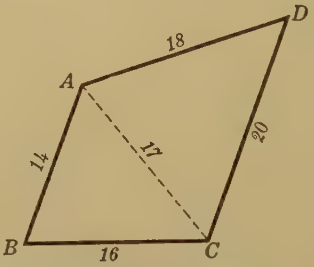
8. Two circles, C_1 and C_2, are tangent externally and each is tangent internally to the circle C_3. If the radii of the circles are 4, 5, and 12, find the angles made by their lines of centers.



9. The two parallel sides of a trapezoid are 12 and 16 and the non-parallel sides are 8 and 10. Find the angles of the trapezoid and also its area.

Suggestion: Draw $DB' \| CB$ and study $\triangle AB'D$. Why is the angle at B' obtuse?

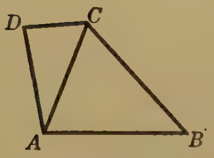


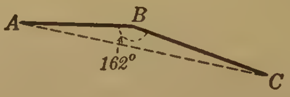


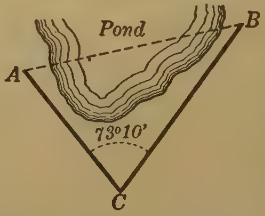
10. In a quadrilateral $ABCD$, $AB = 14$, $BC = 16$, $CD = 20$, $DA = 18$, and the diagonal $AC = 17$. Find the angles of the quadrilateral and also its area.

GROUP B

1. A surveyor wishing to measure a four-sided plot of ground, and having no surveying instrument with him except a tape for measuring distances, measured the four sides and a diagonal and found these to be $AB = 243.2$ feet, $BC = 251.1$ feet, $CD = 89.7$ feet, $DA = 180.3$ feet, and $AC = 195.2$ feet. Find the four angles of the plot $ABCD$. What is the length of the diagonal BD?







2. A road runs from A to B and from B to C, making an angle of 162° at B. If $AB = 1{,}860$ feet and $BC = 2{,}142$ feet, find the length of a straight road from A to C.

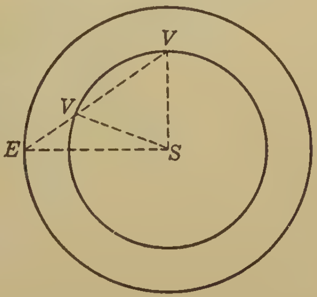
3. To find the distance across a pond from A to B the straight lines AC and BC are measured and found to be 486.2 feet and 591.7 feet, respectively, and the angle C is 73° 10′. Compute the length of AB.

4. In running a straight line in a survey a point is observed to be 21° 30′ to the right of the line, and after going 3,652 feet farther along this line the point is 67° 30′ to the right of it. Find the shortest distance from this point to the line.

5. The diagonals of a parallelogram are 49.2 feet and 61.8 feet and meet in angle of 77° 50′. Find the length of the sides of the parallelogram and also its area.

6. Two stations connected by a straight railway are known to be $2\frac{3}{4}$ miles apart. From a farmhouse it is observed that the sound of the whistle from one station reaches it in 8 seconds and from the other in 14 seconds. What is the shortest distance from the farmhouse to the railway line? (See Ex. 6, p. 85.)

7. The distances in miles from the earth and Venus to the sun are about 93,000,000 and 67,000,000. How far is Venus from us when she appears at a distance of 34° from the sun? Note that two answers are possible. Give both of them.



8. The sides of a triangle ABC are $AB = 13$, $BC = 14$, $CA = 16$. The line AD bisects the side BC at D. Find the angles of the triangle ABD.

9. In the triangle ABC, $AB = 31$, $AC = 42$, and $\angle A = 47°\ 15'$. Find the length of the median from B upon the side AC.

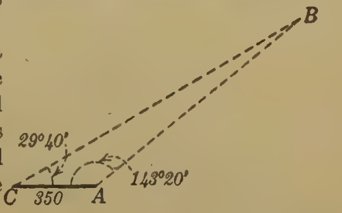
10. A tree stands at a distance from a straight road. At a point A in the road the line to the tree makes an angle of 21° with the line of the road, and at a point B the line to the tree makes an angle of 51° with the line of the road. If AB = 3,400 feet, find the shortest distance from the tree to the road.

Suggestion: Show that as stated this problem has two correct answers and find both.

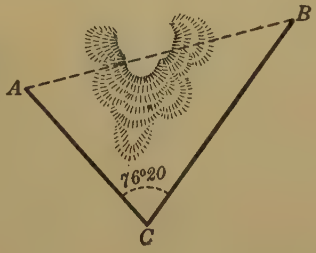
Group C

1. Three intersecting streets form a plot of ground whose sides are 316.7 feet, 421.9 feet, and 472.3 feet. Find the angles of this piece and also its area.

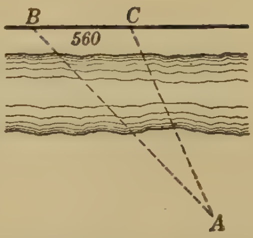
2. A telephone wire is to be strung up a hill of uniform slope from A to B. A distance AC = 350 feet is measured on level ground directly away from the hill, and the angles ACB and CAB are found to be 29° 40′ and 143° 20′, respectively. Find the length of the line from A to B.



3. In making a survey it becomes necessary to find the distance from A to B which are located on opposite sides of a steep hill. Instead of measuring directly across the hill, the lines AC and CB and the angle C are measured and found to be 296 feet, 387 feet, and 76° 20′, respectively. Compute the distance from A to B.



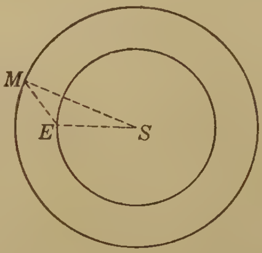
4. It is desired to find the shortest distance from a point A on the bank of a river to a straight road on the opposite side. On this road two points, B and C, are located 560 feet apart and the angles ABC and BCA are found to be 45° 20′ and 113° 50′, respectively. From these data compute the required distance.



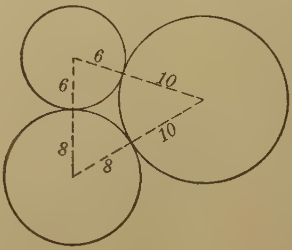
5. The diagonals of a parallelogram are 216.2 feet and 193.7 feet and meet in an angle of 114° 30′. Find the sides of the parallelogram and also its area.

6. From a point of observation, A, two batteries, at B and C, subtend an angle of 81° 15′. The sound from the battery at B reaches the observer $3\frac{1}{2}$ seconds after the flash and the report from that at C reaches him in 7 seconds after the flash. If sound travels 1,090 feet per second, how far apart are the batteries?

7. The distances in miles from the earth and Mars to the sun are about 93,000,000 and 141,000,000, respectively. How far is Mars from us when she appears at a distance of 164° from the sun?

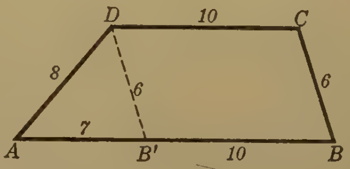


8. Three circles whose radii are 6, 8, 10 are tangent externally as shown in the figure. Find the angles between the lines of centers.



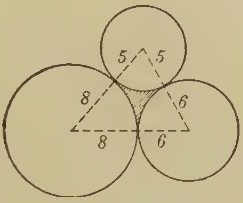
9. The two parallel sides of a trapezoid are 10 and 17, and the non-parallel sides are 6 and 8. Find the angles of the trapezoid and also its area.

Suggestion: As indicated in the figure draw DB' parallel to CB and solve the triangle $AB'D$. Is $\angle AB'D$ acute or obtuse?

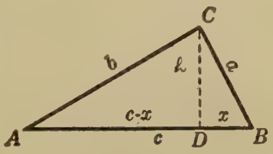


10. The diagonals AC and BD of a quadrilateral meet in a point O, making an angle of 67° 30′. If $AO = 12$, $OC = 17$, $BO = 14$, and $OD = 19$, find the area of the quadrilateral.

MISCELLANEOUS EXERCISES



1. Three circles with radii 5, 6, and 8 are tangent externally as shown in the figure. Find the area (shaded in the figure) inclosed by them.

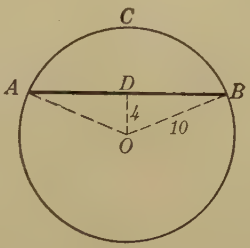


✓2. In the triangle ABC, $\angle A$, $\angle B$, and side c are given. Show how to solve this triangle completely by means of the formulas on right triangles. In what ways is this solution inferior to the solution by means of the sine formula?

Suggestion: In the figure $\frac{c-x}{h} = \cot A$, $\frac{x}{h} = \cot B$.

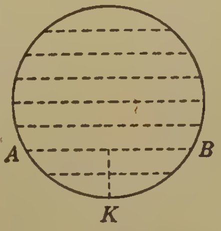
3. In the triangle ABC, $\angle A$, and sides b and c are given. Solve the triangle completely by means of formulas on right triangles. In what ways is this solution inferior to the solution by means of the tangent formula?

Suggestion: Using the figure of Exercise 2, $\triangle ADC$ can be solved at once and this gives the values of h and x.



4. In a circle with radius 10 a chord AB is a distance 4 from the center. Find the area of the segment inclosed by this chord and the arc ACB.

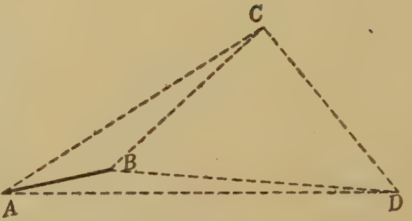
Suggestion: Find: (a) $\angle AOB$, (b) area $AOBC$, (c) area $\triangle AOB$.



5. It is desired to keep a "gauge" on a large cylindrical tank to show the amount of liquid in it when it is filled to various heights. The gauge consists of a series of horizontal lines on one end of the tank, with figures on each line showing how many gallons it contains when filled up to this line. In a tank 8 feet in diameter and 32 feet long how many gallons are contained below a line AB which is 2 feet from the lower edge? One gallon contains 231 cubic inches.

Suggestion: Find the area of the segment AKB and multiply by the length of the tank.

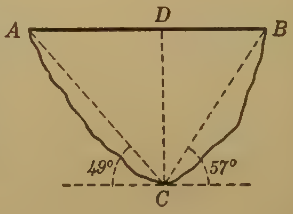
6. Make a table for the gauge for the tank in Ex. 5, the distance between the lines being 12 inches. After completing the lower half of the scale, devise a simple plan for making the upper half of it.



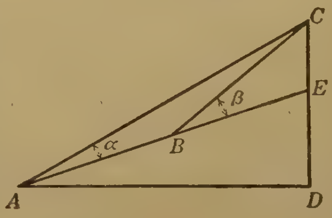
7. Two mountain peaks, C and D, are visible from the campus of a university. It is desired to find the distance from a point A to each of the peaks and also the distances between the peaks. A line AB is measured and found to be 6840 feet. Further, $\angle BAC = 27° \ 40'$; $\angle ABC = 149° \ 50'$; $\angle BAD = 46° \ 30'$; and $\angle ABD = 131° \ 10'$. Find AC, AD, and CD. To check the solution find BC, BD, and CD.

NOTE. While the points C and D are above the level of the horizontal line AB, the surveying is done as if they were in the same horizontal plane with it. The angles that are measured lie in this horizontal plane, and the distances found in the computation are horizontal distances. What is wanted in land surveying and in geographical surveying is horizontal distances. A map shows distances between points as if they were all on the same level.

8. Viewed from a point directly south of it, the top of a hill shows an elevation of 27° 40′. After going along a horizontal road in a due westerly direction for a distance of 1000 feet the top of the hill shows an elevation of 21° 10′. Find height of the hill above the road.



9. A bridge across a deep ravine has its ends at the points A and B. At a point C in the bottom of the ravine it is desired to build a pier to support the bridge. Find the height of this pier if the elevations of A and B as seen from C are 49° and 57°, respectively, and if the length of the bridge (AB) is 567 feet.



10. In surveying for a road the following problem occurs: In the figure the line AD is horizontal and the line DC is vertical. The surveyor measures the angle of elevation of C as seen from A (DAC), the angles α and β, and the length of the line AB. The problem is to compute the line EC. Give the series of steps required in this computation.

11. If in Ex. 10, $DAC = 14° \ 45'$, $\alpha = 4° \ 25'$, $\beta = 7° \ 30'$, and $AB = 1586$ feet, what is the length of EC?

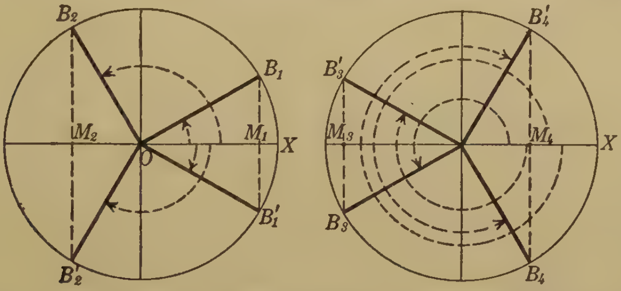
NOTE. Exercises on surveying, page 155, may be taken at this time if desired.

CHAPTER IX

VARIATION OF TRIGONOMETRIC FUNCTIONS

THE study of trigonometric functions as undertaken up to this point is quite sufficient for their use in solving triangles. We will now make a further study of these functions, partly to obtain a clearer and more comprehensive understanding of the functions themselves, and partly to obtain formulas that are necessary in later courses in mathematics. This study will also throw further light on topics that have gone before. In particular, some of the theorems already proved for special angles (see §§88–99) will be extended so as to apply to any angles whatever.

117. Relations of functions of positive and negative angles.—To study the sine and cosine of negative angles construct angles XOB_1, XOB_2, XOB_3, XOB_4, all positive and lying in the different quadrants as shown in the figures. Construct also XOB_1', XOB_2', XOB_3', XOB_4', all negative so that XOB_1 and XOB_1', XOB_2 and XOB_2', etc., are equal numerically in pairs.



From the figures it is evident that

$$\sin XOB_1 = -\sin XOB_1', \ \sin XOB_2 = -\sin XOB_2', \text{ etc.}$$

Hence for all values of A less than 360°

$$\sin A = -\sin(-A) \qquad [52]$$

Since by §85 for any angle A, $\sin A = \sin(A \pm 360°)$ it follows that [52] holds for any angle whatever.

From the figures it is also evident that

$$\cos XOB_1 = \cos XOB_1', \ \cos XOB_2 = \cos XOB_2', \text{ etc.}$$

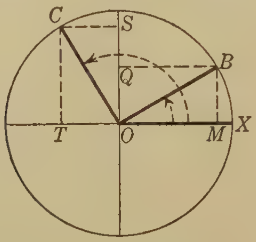
Hence we have, by proceeding as above, that for any angle A

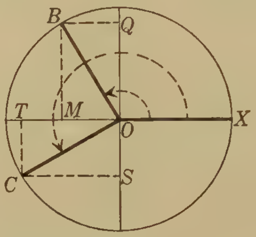
$$\cos A = \cos(-A) \qquad [53]$$

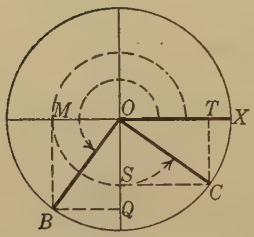
118. Sin $(A \pm 90°)$, cos $(A \pm 90°)$.—In extending the proof of the formulas in §§87–93 the following formulas will be used.

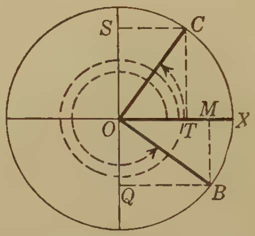
$\sin(A + 90°) = \cos A$ [54] $\quad \cos(A + 90°) = -\sin A$ [56]

$\sin(A - 90°) = -\cos A$ [55] $\quad \cos(A - 90°) = \sin A$ [57]









In each of the four figures XOB represents an angle A, and XOC represents $A + 90°$.

It is easily verified that in each figure $TC = QB$ and $SC = -MB$. Hence,

$$\sin(A + 90°) = \cos A \text{ and } \cos(A + 90°) = -\sin A.$$

Since by §85, $\sin A = \sin(A + 360°)$ and $\cos A = \cos(A + 360°)$, [54] and [56] are proved for any positive angle A.

By noticing that in each figure the angle A is coterminal with a negative angle, it follows that these formulas hold also for negative angles.

Formula [54] may be read: "The sine of an angle is equal to the cosine of an angle which is 90° smaller." If this angle is represented by B then $\sin B = \cos(B - 90°)$ which is [57].

In a similar manner [55] follows from [56].

Hence formulas [54], . . . [57], have been proved for all possible values of A.

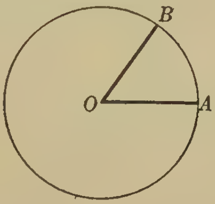
Using [54] and [56] again it follows at once that

$\sin(A + 180°) = -\sin A$ [58] $\quad \cos(A + 180°) = -\cos A$ [59]

That is, adding 180° to the angle (or subtracting 180° from it) changes the sign of both sine and cosine.

119. Radian measure.—Up to this point the degree, minute, and second have been used as units of measure of angles. These units are used in nearly all tables of trigonometric functions. For certain purposes however another unit of angle, called the radian, is used exclusively.

If the vertex of the angle AOB is the center of a circle and if the length of the arc AB is equal to the radius of the circle, then the angle AOB is one **radian.** That is,



A radian is a central angle whose intercepted arc is equal in length to the radius of the circle.

From geometry we know that the circumference of a circle is equal to the diameter multiplied by π where $\pi = 3.1416$ (approximately), or, what is the same thing, that the radius multiplied by π is equal to the semicircumference.

Since one straight angle, which is measured by the semicircumference, contains 180°, it follows that one radian is equal to $180 \div \pi$ degrees. If 3.141593 is taken as the value of π, then the radian is found to be 57.29577 degrees or 57° 17′ 44.8″. In practice one radian is often regarded as 57.3°.

Since 180° is equal to π radians it follows that

$$90^\circ = \frac{\pi}{2},\ 60^\circ = \frac{\pi}{3},\ 30^\circ = \frac{\pi}{6},\ 120^\circ = \frac{2}{3}\pi,\ 135^\circ = \frac{3}{4}\pi,\ 150^\circ = \frac{5}{6}\pi.$$

The second member in each of these equations is expressed in radians. From now on either the radian or the degree will be used as a unit of angle as convenience requires.

120. Rule for converting radians into degrees and degrees into radians.—Since one radian contains $\frac{180}{\pi}$ degrees, it follows that an angle of x radians contains $\frac{180}{\pi}\cdot x$ degrees, and an angle of y degrees contains $y \div \frac{180}{\pi}$, or $\frac{\pi}{180}\cdot y$, radians.

Hence we have the rule:

To convert radians into degrees multiply by 180/π and to convert degrees into radians multiply by π/180.

EXERCISES

1. How many radians are there in 75°? in 84°? in 97°? in 114° 30′? Give each result correct to three decimals.

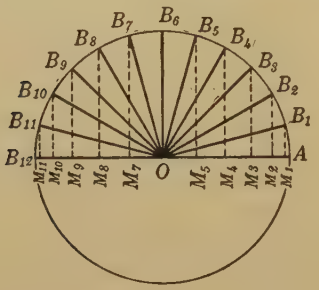
2. How many degrees, minutes, and seconds are there in ½ radian? in 1⅓ radians? in 2 radians? in 2.7 radians?

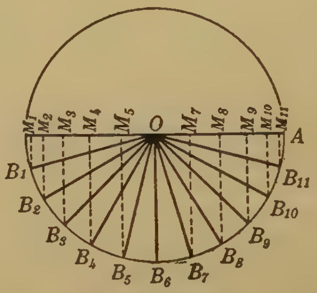
121. Variation of the sine.—From the table on page 25 we have

$\sin 0 = 0.0000$
$\sin 15 = 0.2588$
$\sin 30 = 0.5000$
$\sin 45 = 0.7071$
$\sin 60 = 0.8660$
$\sin 75 = 0.9659$
$\sin 90 = 1.0000$

which shows that as the angle increases from 0° to 90° the sine increases from 0 to 1, but that it increases more slowly as the angle grows larger.

From the formula $\sin A = \sin (180° - A)$ (see §87) we see that as the angle increases from 90° to 180° the sine decreases from 1 to 0 and that the decrease is more rapid as the angle grows toward 180°.





The variation of the sine as the angle varies through the first and second quadrants is also seen in the first figure in which $\angle AOB_1 = 15°$, $\angle AOB_2 = 30°$, etc. If the radius of the circle is taken as the unit of length, then the lengths of M_1B_1, M_2B_2, . . . represent the sines of these angles.

In the second figure the variation of the sine is shown as the angle varies through the third and fourth quadrants. It is evident that in the third quadrant the sine varies from 0 to -1 and in the fourth quadrant from -1 to 0.

The variation of the sine in the third and fourth quadrants as compared with that in the first and second quadrants is expressed by the relation

$$\sin (180° + A) = -\sin A \qquad [58]$$

The variation of the sine of negative angles follows from the relation $\sin (-x) = -\sin x$ [52].

That is, as x varies from 0 to $-90°$, $\sin x$ varies from 0 to -1 in precisely the same manner that it varies from 0 to 1 as x varies from 0 to 90°, and so on.

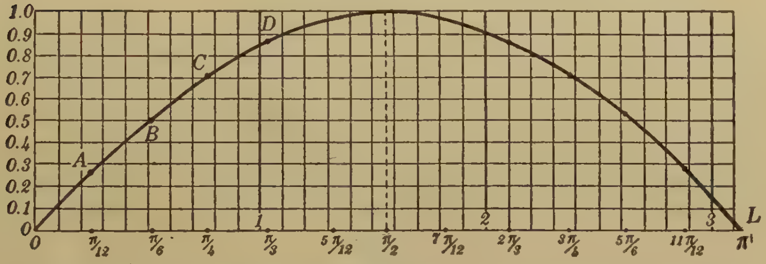
122. Graphic representation of the variation of the sine.— For the purpose of constructing graphs representing the variation of trigonometric functions the radian is used as the unit of angle. We note that

$$15° = \frac{1}{12}\pi,\ 30° = \frac{1}{6}\pi,\ 45° = \frac{1}{4}\pi,\ 60° = \frac{1}{3}\pi,\ 75° = \frac{5}{12}\pi,\ 90° = \frac{1}{2}\pi,$$

$$105° = \frac{7}{12}\pi,\ 120° = \frac{2}{3}\pi;\ 135° = \frac{3}{4}\pi,\ 150° = \frac{5}{6}\pi,\ 165° = \frac{11}{12}\pi,\ 180° = \pi$$

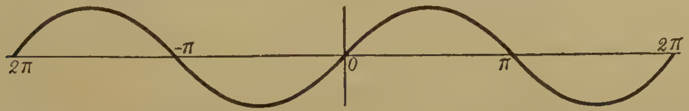
Reducing these values to decimals correct to two places we have,

15° = 0.26 (radians) 60° = 1.05 105° = 1.83 150° = 2.62
30° = 0.52 75° = 1.31 120° = 2.09 165° = 2.88
45° = 0.79 90° = 1.57 135° = 2.36 180° = 3.14



If we plot the points whose abscissas represent angles and whose ordinates represent the corresponding values of the sine, we have the points O, A, B, L, as shown in the figure. The coördinates of these points are O (0, 0), A (.26, .26), B (.52, .50), C (.79, .71), . . . L (3.14, 0). If a smooth curve is drawn through these points, we have the *sine curve* from 0 to π.

If this curve is continued from π to 2π it will have exactly the same shape as the arc shown above, but will lie below the x-axis. Similarly, from 0 to $-\pi$ it will lie below the x-axis, while from $-\pi$ to -2π it will be above the x-axis. The general shape of this curve is shown below.



In this graph the arcs 0 to π, π to 2π, 0 to $-\pi$ and $-\pi$ to -2π have exactly the same shape.

EXERCISES

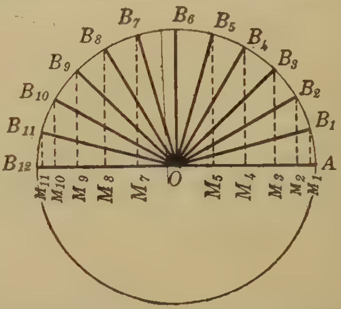
1. From the first figure find sin 10°, sin 40°, sin 70° correct to two decimals and compare with the values given on page 25.

Suggestion: $10° = \frac{\pi}{18} = .17$ (to the nearest hundredth).

2. From the second figure verify that $\sin x = -\sin(-x)$. In particular what are the values of $\sin\frac{\pi}{2}$ and $\sin\left(-\frac{\pi}{2}\right)$? of $\sin\frac{3\pi}{2}$ and $\sin\left(-\frac{3\pi}{2}\right)$?

123. Variation of the cosine.—From the table on page 25 we see that as the angle varies from 0 to 90° the cosine varies from 1 to 0.

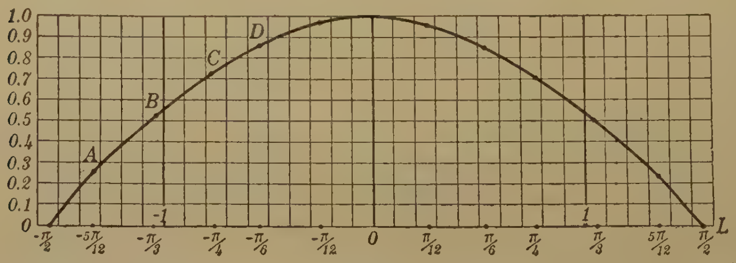
In the figure $\angle AOB_1 = 15°$, $\angle AOB_2 = 30°$, etc. If the radius of the circle is taken as unit of length, then the lengths of OM_1, OM_2, . . . represent the cosines of these angles. In the second quadrant the cosines are OM_7, OM_8, . . . , all of which are negative. It is evident that as the angles vary from 90° to 180°, the cosine varies from 0 to -1.



If as on page 94 we construct a similar figure showing the cosines of angles in the third and fourth quadrants, we find that in the third quadrant the cosine varies from -1 to 0 and in the fourth quadrant it varies from 0 to 1.

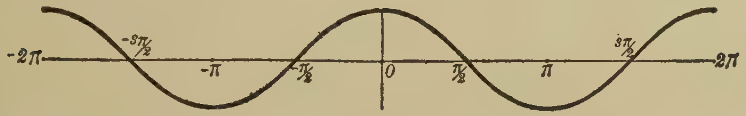
The variation of the cosine for negative angles is shown by the relation $\cos(-x) = \cos x$ [53].

124. Graphic representation of the variation of the cosine.—Starting at the point $-\frac{\pi}{2}$ (-1.57) on the x-axis, we mark the points -1.31, -1.05, -0.79, -0.52, -0.26, 0, 0.26, 0.52, 0.79, 1.05, 1.31, 1.57, which represent the angles $-90°$, $-75°$, $-60°$, $-45°$, $-30°$, $-15°$, 0, 15°, 30°, 45°, 60°, 75°, 90° (see page 95). At these points we erect perpendiculars equal to the cosines of these angles and mark the ends of these A, B, C, . . . , L.



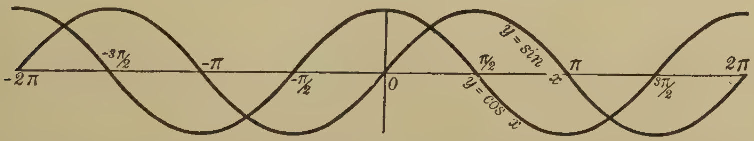
Drawing a smooth curve through these points, we have the *cosine curve* from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

If this curve is continued from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ it will have exactly the same shape as the arc just shown, but will be below the x-axis, while from $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$ it will be above the axis. Similarly from $-\frac{\pi}{2}$ to $-\frac{3\pi}{2}$ it will be below the axis and from $-\frac{3\pi}{2}$ to $-\frac{5\pi}{2}$ it will be above the axis.



From this curve we may verify that $\cos(-x) = \cos x$.

125. Comparison of the sine curve and the cosine curve.—By placing the two graphs on the same axes their relation becomes apparent.



These curves have exactly the same shape, and if the cosine curve is moved a distance $\frac{\pi}{2}$ to the right (or the sine curve a distance $\frac{\pi}{2}$ to the left) the curves will coincide completely.

From this graph formulas [54], . . . [57] (page 92) may now be verified. Thus $\sin(x + 90°) = \cos x$ is verified by the fact that if the sine curve is moved a distance $\frac{\pi}{2}$, or 90°, to the left it will coincide with the cosine curve.

EXERCISES

1. Using the sine curve, verify that $\sin(\pi - x) = \sin x$. (See § 87.)

2. Using the cosine curve, verify that $\cos(\pi - x) = -\cos x$. (See § 87.)

3. Using the sine and cosine curves, verify that if two angles are complementary the sine of either is equal to the cosine of the other.

4. Express each of the following as a function of an angle between 0° and 45°:

(*a*) sin 68°
(*b*) cos 79°
(*c*) sin 100°
(*d*) cos 100°
(*e*) sin 135°
(*f*) cos 169°
(*g*) sin (– 30°)
(*h*) cos (– 30°)
(*i*) sin (– 82°)
(*j*) cos (– 76°)
(*k*) sin (– 120°)
(*l*) cos (– 150°)

126. Variation of the tangent.—The table on page 25 shows that as the angle varies from 1° to 89° the tangent varies from 0.0175 to 57.2900. Since $\tan x = \frac{\sin x}{\cos x}$, it follows that tan 90 becomes $\frac{\sin 90°}{\cos 90°} = \frac{1}{0}$, which does not represent any number whatever, since division by zero is impossible.[1]

It is easily seen that as the angle is made to vary toward 90° the tangent grows large beyond any definite number whatever. Thus $\tan 89°30' = 114.589$, $\tan 89°50' = 343.774$, $\tan 89°59' = 3{,}437.75$, and $\tan 89°59'59'' = 206{,}264.8$.

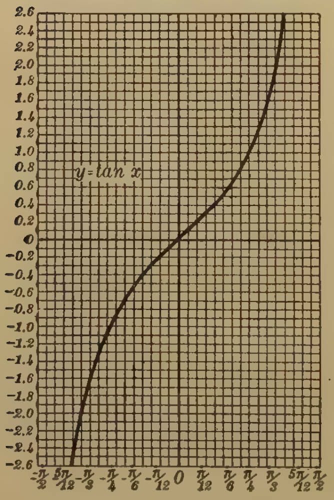
It is customary to say that tan 90° is *infinite*[2] and to write

$$\tan 90° = \infty$$

This means that the angle 90° has no tangent, but that as the angle approaches 90° *the tangent grows large beyond all bounds.*

The graph at the left shows the variation of the tangent as the angle varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. As the angle grows larger toward $\frac{\pi}{2}$ the tangent grows large without bound in the positive direction, and as the angle grows in the negative direction toward $-\frac{\pi}{2}$ the tangent grows without bound in the negative direction. This shows the meaning of the statement:

"As the angle varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ the tangent varies from $-\infty$ to $+\infty$."



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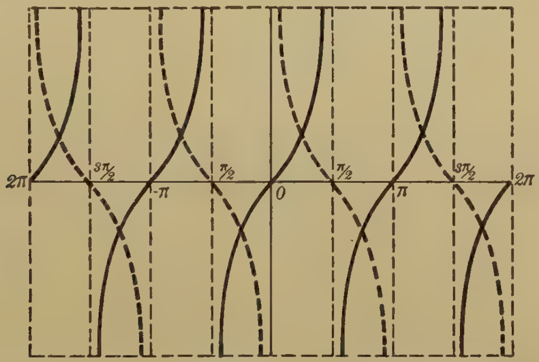
[1] That division by zero is impossible follows directly from the definition of division. "To divide a by b is to find a number c such that $bc = a$. If $b = 0$ then $bc = 0$ for all values of c. Hence if $a \neq 0$ and $b = 0$, there is no number c such that $a \div b = c$. If both a and b are zero, then for *any* value of c, $bc = a$. Hence if $a = 0$ and $b = 0$, $a \div b$ may represent any number whatever. That is, it does not represent any definite number at all.

[2] The word *infinite* means literally *without end*, or *without boundary*, and this is the meaning with which it is used here.

From the relations $\tan x = \dfrac{\sin x}{\cos x}$ and $\sin(\pi + x) = -\sin x$ [58] and $\cos(\pi + x) = -\cos x$ [59] we have

$$\tan(\pi + x) = \tan x \qquad [60]$$

Hence as x varies from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ or from $-\frac{3\pi}{2}$ to $-\frac{\pi}{2}$ the tangent passes through the same variation that it does when x varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. This relation may be verified from the graph below, in which the solid curves represent $\tan x$ and the dotted curves represent $\cot x$.



127. Variation of the cotangent.—From the relation

$$\cot x = \frac{\cos x}{\sin x}$$

we have, as above

$$\cot(\pi + x) = \cot x \qquad [61]$$

This relation may be verified from the graph above.

EXERCISES

1. Using the tangent curve, verify that $\tan(\pi - x) = -\tan x$.

2. Using the cotangent curve, verify that $\cot(\pi - x) = -\cot x$.

3. Using the tangent and cotangent curves, verify $\tan x = \tan(x \pm \pi)$ and $\cot x = \cot(x \pm \pi)$.

4. Express each of the following as a function of an angle between 0° and 45°.

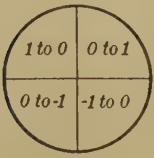
(*a*) tan 69°
(*b*) cot 74°
(*c*) tan 98°
(*d*) cot 112°
(*e*) tan 172°
(*f*) tan (– 40°)
(*g*) cot (– 60°)
(*h*) cot (– 84°)
(*i*) tan (– 94°)
(*j*) tan (– 145°)
(*k*) cot (– 160°)
(*l*) cot (– 174°)

128. Résumé of variation of functions.—We have already seen (§121) that as the angle varies through the four quadrants, the sine varies from 0 to 1 in the first quadrant, then from 1 to -1 in the second and third quadrants, and finally from -1 to 0 in the fourth quadrant.

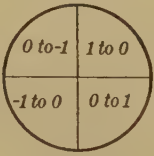
Similarly, we have seen (§123) that for the same variation of the angle the cosine varies from 1 to -1 in the first and second quadrants and from -1 to $+1$ in the third and fourth quadrants. In the first quadrant the tangent varies from 0 to $+\infty$, in the second quadrant from $-\infty$ to 0, in the third from 0 to $+\infty$, and in the fourth from $-\infty$ to 0; while the cotangent varies from $+\infty$ to 0 in the first quadrant, from 0 to $-\infty$ in the second, from $+\infty$ to 0 in the third, and from 0 to $-\infty$ in the fourth.

These variations are most clearly apparent from the graphs on pages 95, 96, and 99.

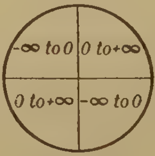
The variations of the secant and cosecant have not been considered in detail, partly because they are less important and partly because they can be derived directly from the relations $\sec x = 1/\cos x$ and $\csc x = 1/\sin x$ when the variations of the sine and cosine are known.



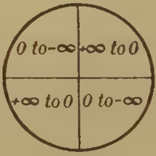
Sin x



Cos x



Tan x

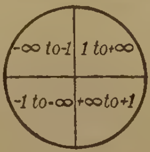


Cot x

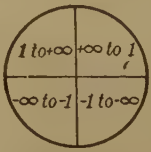
The variations of the four functions are shown schematically in the circles, each of which is divided into four quadrants.

From $\sec x = 1/\cos x$ it follows that as $\cos x$ varies from 1 to 0 in the first quadrant, $\sec x$ varies from 1 to $+\infty$. For the same reason the secant varies from $-\infty$ to -1 in the second quadrant, from -1 to $-\infty$ in the third, and from $+\infty$ to 1 in the fourth.

In precisely the same manner it follows from $\csc x = 1/\sin x$ and the known variation of the sine that the cosecant varies from $+\infty$ to 1 in the first quadrant, from 1 to $+\infty$ in the second, from $-\infty$ to -1 in the third, and from -1 to $-\infty$ in the fourth.

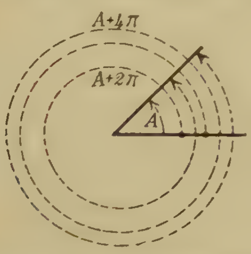


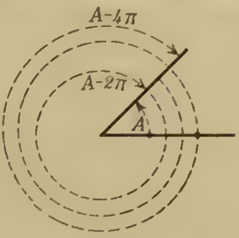
Sec x



Cosec x

129. Periodicity of functions.—It is at once obvious from the definitions of the trigonometric functions that adding 2π to an angle or subtracting 2π from it does not change the value of any of its functions. It follows that the variation of each function is exactly the same when the angle varies from 2π to 4π as it is when the angle varies from 0 to 2π, and that the variation is also the same when the angle varies from -2π to 0.





It is evident that the same variation is repeated as the angle varies from 4π to 6π, from 6π to 8π, etc., and also as it varies from -4π to -2π, from -6π to -4π, etc. More generally, if the x-axis, the axis on which the angles are laid off, is divided into equal parts, each 2π in length, all the trigonometric functions will repeat their variation on each of these divisions.

This is indicated in mathematical language by saying that these functions have a **period** of 2π.

By looking at the graph on page 99 it is further evident that the tangent and cotangent have a smaller period, namely a period of π. That is, the tangent, for instance, goes through a certain variation as the angle varies from 0 to π, and then repeats exactly the same variation as the angle varies from π to 2π, from 2π to 3π, etc., and also from $-\pi$ to 0, from -2π to $-\pi$, etc.

Similar remarks apply to the cotangent. Hence both tangent and cotangent have a smallest period π. It is clear that this does not contradict the statement above that these functions also have a period 2π. For the sine and cosine, and consequently also for the secant and cosecant, the smallest period is 2π.

EXERCISES

1. Show that a curve representing the variation of the sine in the first quadrant may be used as a model for constructing the whole sine curve.

2. Show how curves representing the variations of the cosine, the tangent, and the cotangent in the first quadrant may be used as models in constructing the complete curves for these functions.

130. Functions of large angles.—The tables give the functions of angles from 0 to 90°, and in some cases only from 0 to 45°. Hence it is of practical importance to find the values of functions of large angles in terms of functions of small angles. This problem we have already solved in part:

(a) By means of the theorem (§8), that "the cofunctions of complementary angles are equal," functions of angles between 45° and 90° may always be found in terms of functions of angles between 0 and 45°.

(b) By means of the theorem (§87), that "if two angles are supplementary their sines and also their cosecants are equal, while their cosines, tangents, cotangents, and secants are equal numerically, but opposite in sign," functions of angles between 90° and 180° may be found in terms of functions of angles between 0 and 90°, and then by the use of (a) these may be found in terms of functions of angles between 0 and 45°.

We will now study this problem more generally. From the periodicity of trigonometric functions it follows that in the case of the sine, cosine, secant, and cosecant any multiple of 2π, which includes 2π, $2 \cdot 2\pi$, $3 \cdot 2\pi$, etc. (that is, any multiple of 360°), may be added or subtracted without changing the value of the functions. This enables us to reduce any of these functions of any angle to the same function of an angle between 0 and 360°.

Since the tangent and cotangent have a period π, any multiple of π (that is, any multiple of 180°) may be added or subtracted without changing the value of the functions. This enables us to reduce either of these functions of any angle to the same function of an angle between 0 and 180°.

It remains to reduce the sines and cosines (and in consequence the secants and cosecants) of angles in the third and fourth quadrants to functions of angles in the first or second quadrants.

This is done by means of the formulas

$$\sin(x \pm \pi) = -\sin x \quad [58] \quad \text{and} \quad \cos(x \pm \pi) = -\cos x \quad [59]$$

We will now attempt to collect these various formulas and if possible reduce them to a simpler form for practical use and for more easy memorizing. At best the result is somewhat complicated.

131. The five-part theorem.—The following is a summary of the various steps that may be used in reducing a function of any angle to a function of an angle between 0 and 45°.

(a) *The tangent or cotangent of any angle (positive or negative) may be reduced to the tangent or cotangent of an angle between* 0° *and* 180° *by adding to or subtracting from the given angle a multiple of* 180°. (§129.)

(b) *The sine, cosine, secant, or cosecant of any angle (positive or negative) may be reduced to the same function of an angle between* 0 *and* 360° *by adding to or subtracting from the given angle a multiple of* 360°. (§129.)

(c) *The sine, cosine, secant, or cosecant of an angle between* 180° *and* 360° *may be reduced to the same function of an angle between* 0 *and* 180° *by subtracting* 180° *from the angle and changing the sign of the function.* (§118).

(d) *A function of an angle between* 90° *and* 180° *may be reduced to the same function of an angle between* 0 *and* 90° *by subtracting the given angle from* 180°, *and changing the sign of the function, except in the case of the sine and cosecant which retain the same sign.* (§87.)

(e) *A function of an angle between* 45° *and* 90° *may be reduced to a function of an angle between* 0 *and* 45° *by subtracting the given angle from* 90° *and taking the cofunction of the result.* (§8.)

The method of using this theorem is shown in the following:

Example 1. Reduce $\tan(1248°)$ to a function of an angle between 0° and 45°.

Solution: By (a) $\tan 1248° = \tan(1248° - 6 \cdot 180°) = \tan 168°$.
By (d) $\tan 168° = -\tan(180° - 168°) = -\tan 12°$.

Example 2. Reduce $\sin(-794°)$ to a function of an angle between 0° and 45°.

Solution: By (b) $\sin(-794°) = \sin(3 \cdot 360° - 794°) = \sin 286°$.
By (c) $\sin 286° = -\sin(286° - 180°) = -\sin 106°$.
By (d) $-\sin 106° = -\sin(180° - 106°) = -\sin 74°$.
By (e) $-\sin 74° = -\cos(90° - 74°) = -\cos 16°$.

EXERCISES

Reduce each of the following to a function of an angle between 0 and 45°. Watch for opportunities to use short methods.

1. $\sin 317°$
2. $\cos 408°$
3. $\tan 346°$
4. $\cot 594°$
5. $\sec 493°$
6. $\csc 786°$
7. $\sin(-812°)$
8. $\cos(-214°)$
9. $\cot(-987°)$
10. $\tan(-1260°)$
11. $\sec(-397°)$
12. $\csc(-418°)$
13. $\sin(-1298°)$
14. $\sin 1298°$
15. $\cos(-1785°)$
16. $\cos 1785°$
17. $\tan 894°$
18. $\tan(-894°)$
19. $\sin(-564°)$
20. $\cos(-564°)$

21. Show that any function of an even multiple of $90° + x$, that is, of $2n \cdot 90° + x$, is equal to $\pm$ the same function of x, the sign being determined by the quadrant in which the given angle falls.

132. Functions of special angles.—A short table of functions of special angles is given in §11, and functions of other angles are given in §§126–128. We will now arrange a larger table of functions of such angles, including in it all that have been given before.

Func.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°
Sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1
Cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{3}$	-1	0
Tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{3}\sqrt{3}$	0	∞
Cot	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{3}$	-1	$-\sqrt{3}$	∞	0
Sec	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{3}\sqrt{3}$	-1	∞
Csc	∞	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	∞	-1

From the sines and cosines for 30°, 45°, 60° the values for 120°, 135°, 150° are obtained from $\sin(180 - x) = \sin x$ and $\cos(180 - x) = -\cos x$. The values of the other functions may be obtained from the relations

$$\tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{\cos x}{\sin x},\ \sec x = \frac{1}{\cos x},\ \csc x = \frac{1}{\sin x}.$$

The meaning of the special designation for cot 0, tan 90°, csc 0, etc. has already been considered (§126).

Similar remarks apply to the other cases where the values of functions are given as ∞ (infinity). The real fact is that these functions have no values (*do not exist*, or *are not defined*) for these special angles. The values of these functions for 360° are clearly the same as for 0°.

133. Extension of proof of identities.—In §§89–92 the following formulas were proved:

$$\begin{aligned}
\sin(A + B) &= \sin A \cos B + \cos A \sin B\\
\sin(A - B) &= \sin A \cos B - \cos A \sin B\\
\cos(A + B) &= \cos A \cos B - \sin A \sin B\\
\cos(A - B) &= \cos A \cos B + \sin A \sin B
\end{aligned}$$

The proofs were made under the restriction that all angles involved were positive and that the sum of two angles involved should be less than 180°. That is, the formulas were proved for such angles as occur in triangles. It now remains to prove these formulas when A and B represent any angles whatever.

The proof will be made by showing that these formulas hold when either angle is increased or decreased by 90°. If this is proved we shall find that the formulas hold when the given angles are increased or decreased by 90° any number of times and hence for any angles whatever. In the proof we need to make use of the following formulas:

$$\sin(A+90) = \cos A \quad [54] \qquad \cos(A+90) = -\sin A \quad [56]$$
$$\sin(A-90) = -\cos A \quad [55] \qquad \cos(A-90) = \sin A \quad [57]$$

which have been proved for all values of A.

(1) Assuming $\sin(A+B) = \sin A \cos B + \cos A \sin B$, which is known for positive acute angles, we will show that

$$\sin(A+90°+B) = \sin(A+90°)\cos B + \cos(A+90°)\sin B\,. \tag{I}$$

Using [54], we have $\sin(\overline{A+90°+B}) = \sin(A+B+90°) = \cos(A+B)$.
Hence the first member of equation I has been reduced to $\cos(A+B)$.
Using [54] and [56] we have,

$$\sin(A+90°)\cos B + \cos(A+90°)\sin B = \cos A \cos B - \sin A \sin B,$$

which by [20] is equal to $\cos(A+B)$.
Hence each member of I has been reduced to $\cos(A+B)$, and the identity is proved provided A and B are subject to the restrictions imposed in §89.

That is we have shown that the formula for the sine of the sum of two angles holds when either angle is increased by 90°.

(2) Assuming, $\sin(A+B) = \sin A \cos B + \cos A \sin B$, we will prove that

$$\sin(A-90°+B) = \sin(A-90°)\cos B + \cos(A-90°)\sin B. \tag{II}$$

Using [55],

$$\sin(A-90°+B) = \sin(\overline{A+B}-90°) = -\cos(A+B).$$

Using [55] and [57], the second member of II reduces to

$$-\cos A \cos B + \sin A \sin B = -\cos(A+B).$$

It follows as under (1) that the formula for $\sin(A+B)$ has been proved when either A or B is decreased by 90°.
Hence we know that

$$\sin(A_1+B_1) = \sin A_1 \cos B_1 + \cos A_1 \sin B_1$$

where A_1 and B_1 may be either 90° larger or smaller than A and B.
It follows, therefore, that this formula holds for any angles whatever.

It may be proved in exactly the same manner that the other three formulas hold for any angles whatever.

EXERCISES

Prove the following identities:

1. $\sin(\overline{A + 90^\circ} - B) = \sin(A + 90^\circ)\cos B - \cos(A + 90^\circ)\sin B.$
2. $\sin(A - \overline{B + 90^\circ}) = \sin A \cos(B + 90^\circ) - \cos A \sin(B + 90^\circ).$
3. $\sin(A - \overline{B - 90^\circ}) = \sin A \cos(B - 90^\circ) - \cos A \sin(B - 90^\circ).$
4. $\cos(\overline{A + 90^\circ} + B) = \cos(A + 90^\circ)\cos B - \sin(A + 90^\circ)\sin B.$
5. $\cos(\overline{A - 90^\circ} + B) = \cos(A - 90^\circ)\cos B - \sin(A - 90^\circ)\sin B.$
6. $\cos(A - \overline{B + 90^\circ}) = \cos A \cos(B + 90^\circ) + \sin A \sin(B + 90^\circ).$

134. Important trigonometric identities.—Certain important trigonometric identities depend upon the formulas giving the sum of two functions as a product and the converse formulas giving the product of two functions as a sum. Replacing P and Q in the formulas of §98 by A and B we have

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) \qquad [32]$$
$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B) \qquad [33]$$
$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) \qquad [34]$$
$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B) \qquad [35]$$

In §99 we have the formulas

$$\sin A \cos B = \tfrac{1}{2}\sin(A + B) + \tfrac{1}{2}\sin(A - B) \qquad [36]$$
$$\cos A \sin B = \tfrac{1}{2}\sin(A + B) - \tfrac{1}{2}\sin(A - B) \qquad [37]$$
$$\cos A \cos B = \tfrac{1}{2}\cos(A + B) + \tfrac{1}{2}\cos(A - B) \qquad [38]$$
$$\sin A \sin B = -\tfrac{1}{2}\cos(A + B) + \tfrac{1}{2}\cos(A - B) \qquad [39]$$

The use of these formulas is illustrated in the following examples:

Example 1. Prove $\sin^2\theta\cos\theta = \tfrac{1}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta$.

Proof: Using

$$\sin 2\theta = 2\sin\theta\cos\theta, \text{ or } \sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta, \qquad [22]$$
$$\sin^2\theta\cos\theta = \sin\theta\sin\theta\cos\theta = \tfrac{1}{2}\sin\theta\sin 2\theta,$$

which by [39] is equal to

$$\tfrac{1}{4}(-\cos 3\theta + \cos\theta) = \tfrac{1}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta.$$

Example 2. If $A + B + C = 180^\circ$, prove $\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$.

Proof: By [12], since $C = 180^\circ - (A + B)$, $\sin C = \sin(A + B)$, and by [22]

$$\sin(A + B) = 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A + B).$$

By [32]

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A - B).$$

Hence,

$$\sin A + \sin B + \sin C = 2\sin\tfrac{1}{2}(A + B)\left[\cos\tfrac{1}{2}(A - B) + \cos\tfrac{1}{2}(A + B)\right]$$
$$= 4\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}A\cos(-\tfrac{1}{2}B).$$

Since

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}C = 90^\circ,\ \sin\tfrac{1}{2}(A + B) = \cos\tfrac{1}{2}C.$$

Using $\cos(-\tfrac{1}{2}B) = \cos\tfrac{1}{2}B$ [53], we have

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

Example 3. Prove that $\sin^5\theta = \frac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta)$.

Proof: By [24]

$$\begin{aligned}
\sin^5\theta &= (\sin^2\theta)^2\sin\theta = \left(\frac{1-\cos 2\theta}{2}\right)^2\sin\theta \\
&= \tfrac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)\sin\theta \\
&= \tfrac{1}{4}\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\sin\theta \\
&= \tfrac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)\sin\theta \\
&= \tfrac{1}{8}(3\sin\theta - 4\cos 2\theta\sin\theta + \cos 4\theta\sin\theta) \\
&= \tfrac{1}{8}(3\sin\theta - 2\sin 3\theta + 2\sin\theta + \tfrac{1}{2}\sin 5\theta - \\
&\qquad \tfrac{1}{2}\sin 3\theta) \\
&= \tfrac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta).
\end{aligned}$$

Example 4. Express as a product the sum of the series

$S_n = \sin A + \sin(A+B) + \sin(A+2B) + \ldots + \sin[A+(n-1)B]$.

Solution: Multiply each term of the series by $\sin\frac{1}{2}B$.

Then

$$S_n\sin\tfrac{1}{2}B = \sin A\sin\tfrac{1}{2}B + \ldots + \sin[A+(n-1)B]\sin\tfrac{1}{2}B.$$

By [39]

$$\begin{aligned}
\sin A\sin\tfrac{1}{2}B &= \tfrac{1}{2}\cos(A-\tfrac{1}{2}B) - \tfrac{1}{2}\cos(A+\tfrac{1}{2}B), \\
\sin(A+B)\sin\tfrac{1}{2}B &= \tfrac{1}{2}\cos(A+\tfrac{1}{2}B) - \tfrac{1}{2}\cos(A+\tfrac{3}{2}B), \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\sin[A+(n-1)B]\sin\tfrac{1}{2}B & \\
&= \tfrac{1}{2}\cos[A+\tfrac{1}{2}(2n-3)B] - \tfrac{1}{2}\cos[A+\tfrac{1}{2}(2n-1)B].
\end{aligned}$$

Adding, we have,

$$S_n\sin\tfrac{1}{2}B = \tfrac{1}{2}\cos(A-\tfrac{1}{2}B) - \tfrac{1}{2}\cos[A+\tfrac{1}{2}(2n-1)B].$$

Applying [35] to the second member and dividing by $\sin\frac{1}{2}B$, we have

$$S_n = \frac{\sin[A+\frac{1}{2}(n-1)B]\sin\frac{n}{2}B}{\sin\frac{1}{2}B}$$

Example 5. Express as a product the sum of the series

$S_n = \cos A + \cos(A+B) + \cos(A+2B) + \ldots + \cos[A+(n-1)B]$

Solution: Multiply each term of the series by $\sin\frac{1}{2}B$.

Then,

$$S_n\sin\tfrac{1}{2}B = \cos A\sin\tfrac{1}{2}B + \ldots + \cos[A+(n-1)B]\sin\tfrac{1}{2}B.$$

Using [37] and proceeding as in Example 4, we have

$$\begin{aligned}
\cos A\sin\tfrac{1}{2}B &= \tfrac{1}{2}\sin(A+\tfrac{1}{2}B) - \tfrac{1}{2}\sin(A-\tfrac{1}{2}B) \\
\cos(A+B)\sin\tfrac{1}{2}B &= \tfrac{1}{2}\sin(A+\tfrac{3}{2}B) - \tfrac{1}{2}\sin(A+\tfrac{1}{2}B) \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\cos[A+(n-1)B]\sin\tfrac{1}{2}B &= \tfrac{1}{2}\sin\left(A+\frac{2n-1}{2}B\right) - \tfrac{1}{2}\sin\left(A+\frac{2n-3}{2}B\right).
\end{aligned}$$

Adding and using [33],

$$\begin{aligned}
S_n\sin\tfrac{1}{2}B &= -\tfrac{1}{2}\sin(A-\tfrac{1}{2}B) + \tfrac{1}{2}\sin[A+\tfrac{1}{2}(2n-1)B] \\
&= \cos[A+\tfrac{1}{2}(n-1)B]\sin\tfrac{n}{2}B
\end{aligned}$$

Hence,

$$S_n = \frac{\cos[A+\frac{1}{2}(n-1)B]\sin\frac{n}{2}B}{\sin\frac{1}{2}B}.$$

Note that the angles in the series in Examples 4 and 5 are the terms of an arithmetic progression in which the first term is A and the common difference is B. Note also that the method used for finding these sums has certain elements in common with the usual method for finding the sum of a geometric progression. The formulas fail when $\sin \frac{1}{2} B = 0$, that is, when B is a multiple of 2π.

These formulas are important in certain applications in physics and in more advanced courses in mathematics where this subject is greatly extended.

EXERCISES

GROUP A

Express each of the following as a function of an angle between 0° and 90°.

1. $\sin 198°$.
2. $\cos 216°$.
3. $\tan (-416°)$.
4. $\cot (-325°)$.
5. $\sec 417°$.
6. $\csc 914°$.
7. $\sin 426°$.
8. $\cos (-641°)$.
9. $\tan 194°$.
10. $\cot 253°$.
11. $\sec (-714°)$.
12. $\csc 317°$.
13. $\sin (-568°)$.
14. $\cos 617°$.
15. $\tan 562°$.
16. $\cot 832°$.
17. $\sec 312°$.
18. $\csc 794°$.
19. Using the method of page 105, prove $\sin (A - B) = \sin A \cos B - \cos A \sin B$ for all values of A and B.

Prove the following identities:

20. $\sin (A + B) + \sin (A - B) = 2 \sin A \cos B$
21. $\sin (A + B) - \sin (A - B) = 2 \sin B \cos A$
22. $\cos (A + B) + \cos (A - B) = 2 \cos A \cos B$
23. $\cos (A + B) - \cos (A - B) = -2 \sin A \sin B$
24. $\sin 3A = 3 \sin A - 4 \sin^3 A$
25. $\dfrac{\sin A + \cos A}{1 + \sin 2A} = \dfrac{1}{\sin A + \cos A}$
26. $\cos 4A = \cos^4 A - 6 \cos^2 A \sin^2 A + \sin^4 A$
27. $\dfrac{\sin 3A}{\cos A} + \dfrac{\cos 3A}{\sin A} = \dfrac{2}{\tan 2A}$
28. $\dfrac{1 + \cos A \csc A \sin B \sec B}{\cos A \csc A - \sin B \sec B} = \sin (A + B) \sec (A + B)$
29. $\sin A \sec A + \sin B \sec B = \sin (A + B) \sec A \sec B$.

GROUP B

Express each of the following as a function of an angle between 0° and 90°.

1. $\tan 914°$.
2. $\sin 307°$.
3. $\cos (-374°)$.
4. $\cot 394°$.
5. $\sec 274°$.
6. $\cot (-316°)$.
7. $\sec 532°$.
8. $\cos 241°$.
9. $\csc 741°$.
10. $\cot 956°$.
11. $\csc (-168°)$.
12. $\tan (-416°)$.
13. $\sec (-325°)$.
14. $\sin (-216°)$.
15. $\sin 867°$.
16. $\cos 567°$.
17. $\tan 241°$.
18. $\csc 291°$.

19. Using the method of page 105, prove

$\cos(A+B) = \cos A \cos B - \sin A \sin B$, for all values of A and B.

Prove the following identities:

20. $\cos(n-1)A \cos A - \sin(n-1)A \sin A = \cos nA$

21. $2 \sin\left(\frac{\pi}{4} - A\right) \cos\left(\frac{\pi}{4} + B\right) = \overset{\text{sin}}{\cancel{\cos}}(A-B) - \sin(A+B)$

22. $\cos A \csc A + \sin B \sec B = \cos(A-B) \csc A \sec B$

23. $\dfrac{\sin A \sec A \sin B \sec B + 1}{1 - \sin A \sec A \sin B \sec B} = \cos(A-B) \sec(A+B)$

24. $\cos 3A = 4\cos^3 A - 3\cos A$

25. $\dfrac{1+\cos 2A}{1-\cos 2A} = \cot^2 A$

26. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

27. $\dfrac{\cos A - \cos B}{\cos A + \cos B} = -\tan\frac{1}{2}(A+B)\tan\frac{1}{2}(A-B)$

28. $\dfrac{\sin(A+B)}{\cos(A-B)} = \dfrac{\cot A + \cot B}{1 + \cot A \cot B}$

29. $\cos A + \cos 3A + \cos 5A + \cos 7A = 4\cos A \cos 2A \cos 4A$.

GROUP C

Express each of the following as a function of an angle between 0° and 90°:

1. cot 294°.
2. sin 167°.
3. cos (– 264°).
4. tan 316°.
5. sec 572°.
6. csc 127°.
7. tan (– 163°).
8. cot(– 717°).
9. cos 819°.
10. sec (– 275°).
11. sin(– 220°).
12. csc(– 271°).
13. sin 580°.
14. csc 1240°.
15. sec 850°.
16. cot 1064°.
17. cos 240°.
18. tan 420°.

19. Using the method of page 105, prove the formula

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ for all values of A and B.

Prove the following identities:

20. $\sin(3A+2B) + \sin(3A-2B) = 2\sin 3A \cos 2B$.

21. $\sin(6A+B) - \sin(6A-B) = 2\sin B \cos 6A$.

22. $\cos(4A+3B) + \cos(4A-3B) = 2\cos 4A \cos 3B$.

23. $\cos(A+6B) - \cos(A-6B) = -2\sin A \sin 6B$.

24. $\dfrac{\sin(n-2)A + \sin(nA)}{\cos(n-2)A - \cos(nA)} = \cot A$

25. $\dfrac{\cos A \csc A + \cos B \csc B}{\cos A \csc A - \cos B \csc B} = -\sin(A+B)\csc(A-B)$

26. $\cos(A+B) + \sin(A-B) = 2\sin\left(\frac{\pi}{4} + A\right)\cos\left(\frac{\pi}{4} + B\right)$

$= -\left[\cos\left(B+\frac{\pi}{4}\right) + \sin\left(B+\frac{\pi}{4}\right)\right]\left[\cos\left(A+\frac{\pi}{4}\right) - \sin\left(A+\frac{\pi}{4}\right)\right]$

27. $\dfrac{\cos 2A}{1+\sin 2A} = \dfrac{\cos A - \sin A}{\cos A + \sin A}$

28. $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\cos A - \sin A}{\cos A + \sin A} = 2\sin 2A \sec 2A$

EXERCISES

Prove the following identities:

1. $\cot 3A = \dfrac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$
2. $\sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A$
3. $\cos 4A = 1 - 8 \cos^2 A + 8 \cos^4 A$
4. $\tan (A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$
5. $\cot (A + B + C) = \dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot B \cot C + \cot C \cot A - 1}$
6. $\dfrac{1}{1 + \sin^2 A} + \dfrac{1}{1 + \cos^2 A} + \dfrac{1}{1 + \sec^2 A} + \dfrac{1}{1 + \csc^2 A} = 2$
7. $\cos 3A \csc A + \sin 3A \sec A = 2 \cos 2A \csc 2A$
8. $\sin 5A = 16 \sin^5 A - 20 \sin^3 A + 5 \sin A$
9. $\cos 5A = 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$
10. $\tan 4A = \dfrac{4 \tan A\ (1 - \tan^2 A)}{1 - 6 \tan^2 A + \tan^4 A}$

By substituting in the series in examples 4 and 5 on page 107 find the sums of the following series:

11. $\sin A + \sin 2A + \sin 3A + \dots + \sin (n A)$
12. $\cos A + \cos 2A + \cos 3A + \dots + \cos (n A)$
13. $\sin A + \sin 3A + \sin 5A + \dots + \sin (2 n - 1) A$
14. $\cos A + \cos 3A + \cos 5A + \dots + \cos (2 n - 1) A$

If A, B, C are angles of a triangle, prove each of the following:

Suggestion: Since $A + B + C = 180°$, we have, $\sin (A + B) = \sin C$ [12], $\cos (A + B) = -\cos C$ [13], and also $\sin \frac{1}{2} (A + B) = \cos \frac{1}{2} C$ [1].

15. $\sin A + \sin B - \sin C = 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \cos \frac{1}{2} C$
16. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$
17. $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \sin \frac{1}{2} C$
18. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
19. $\sin^2 A + \sin^2 B + \sin^2 C = 2 (1 + \cos A \cos B \cos C)$
20. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$
21. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
22. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$
23. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

CHAPTER X

INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS

In the courses in mathematics which precede trigonometry there occur several cases of what are called inverse operations. Thus subtraction is the inverse operation of addition, division is the inverse of multiplication, and finding a root is the inverse of raising to a power. In the course in trigonometry we have already encountered pairs of operations in which each one is the inverse of the other.

Thus the two operations—(1) finding the sine of a given angle and, (2) finding an angle whose sine is given—are inverse, each of the other. Such operations lead to what are called inverse trigonometric functions which we now proceed to study.

135. Inverse trigonometric functions and their notations.—The sine, cosine, tangent, cotangent, secant, and cosecant of an angle are all definitely fixed when the angle is given, and it is for this reason that they are called functions of the angle.

If an angle is known to be acute and if one of its functions is given, then the angle is definitely fixed. For this reason an angle is said to be a function of its sine, of its cosine, of its tangent, etc.

Thus "the angle whose sine is y" is a function of y. This function of y is designated by the symbol $\sin^{-1} y$, which is read "the angle whose sine is y," or "the inverse sine y," or "arcsin y."

Similarly "the angle whose cosine is y" is designated by the symbol $\cos^{-1} y$, which is also read "the inverse cosine y," or "arccos y." The symbols $\tan^{-1} y$, $\cot^{-1} y$, $\sec^{-1} y$, and $\csc^{-1} y$ are used to designate, respectively, "the angle whose tangent is y," "the angle whose cotangent is y," "the angle whose secant is y," and "the angle whose cosecant is y."

The six functions, $\sin^{-1} y$, $\cos^{-1} y$, $\tan^{-1} y$, $\cot^{-1} y$, $\sec^{-1} y$, $\csc^{-1} y$, are **inverse trigonometric functions** of y.

It should be noted with care that each of these symbols represents an *angle*. Thus the whole expression "$\sin^{-1} y$" represents an angle, and it gives the information that the sine of the angle is y. That is, $y = \sin x$ and $x = \sin^{-1} y$ represent the same fact.

In expressions like $\sin^{-1} y$ the $^{-1}$ is in no sense a negative exponent, but indicates simply an *inverse function*. Thus $(\sin y)^{-1}$ means $\frac{1}{\sin y}$ (one divided by the sine of the angle y), while $\sin^{-1} y$ means the angle whose sine is y.[1]

136. Simple examples involving inverse functions.—As was suggested on the preceding page, many of the examples already solved in this course really involve the use of inverse functions. We will now restate some of these, using the inverse notation.

Example 1. Find $\sin^{-1}\frac{1}{2}$, if it is given that the angle is acute.

Solution: Since it is known that $\sin 30° = \frac{1}{2}$, it follows that $30° = \sin^{-1}\frac{1}{2}$. This result may also be given in radians, namely, $\frac{\pi}{6} = \sin^{-1}\frac{1}{2}$.

Example 2. Find $\cos^{-1}\frac{\sqrt{3}}{3}$ if it is given that the angle is acute.

Solution: $\frac{\sqrt{3}}{3} = .57735.$ From the table of natural functions we find that the angle whose cosine is 0.57735 is 54° 44′.
Hence

$$\cos^{-1}\frac{\sqrt{3}}{3} = 54° 44'.$$

Example 3. Find the value of $\tan\left(\sin^{-1}\frac{2}{3}\right)$ if it is given that the angle is acute.

Solution: This example was solved on page 15, where it was required to find the functions (among them the tangent) of an angle whose sine is $\frac{2}{3}$. We may now carry out the solution as follows:

$$\text{Let } x = \sin^{-1}\frac{2}{3}. \quad \text{Then } \sin x = \frac{2}{3}.$$

Hence,

$$\cos x = \sqrt{1 - \frac{4}{9}} = \frac{1}{3}\sqrt{5}$$

Therefore,

$$\tan x = \frac{\frac{2}{3}}{\frac{1}{3}\sqrt{5}} = \frac{2}{\sqrt{5}}, \text{ or } \tan\left(\sin^{-1}\frac{2}{3}\right) = \frac{2}{\sqrt{5}}.$$

It should be understood clearly that the only new element introduced here in this problem is the inverse notation.

Example 4. Express each of the other five functions of $\sin^{-1} u$ in terms of u.

This example is solved completely on page 16. The results are given in the first column of the table at the bottom of that page, $\sin A$ being used instead of u.

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[1] The symbol $^{-1}$ is used for *any* inverse function out of analogy to the particular case of division where this symbol is a negative exponent as well as an inverse symbol.

EXERCISES

Find the value of x in each of the following, the angle being acute:

1. $x = \sin^{-1} \frac{1}{2}\sqrt{2}$.

2. $x = \tan^{-1} \sqrt{3}$.

3. $x = \cos^{-1} \frac{\sqrt{2}}{2}$.

4. $x = \sin^{-1} \frac{\sqrt{3}}{2}$.

5. $x = \tan^{-1} \frac{\sqrt{3}}{3}$.

6. $x = \cos^{-1} \frac{3}{4}$.

Find the values of the following:

7. $\cos (\sin^{-1} u)$.

8. $\sin (\cos^{-1} u)$.

9. $\tan (\sin^{-1} u)$.

10. $\tan (\cot^{-1} u)$.

11. $\sin (\csc^{-1} u)$.

12. $\cos (\tan^{-1} u)$.

13. $\tan (\cos^{-1} u)$.

14. $\cot (\cos^{-1} u)$.

15. $\sec (\tan^{-1} u)$.

137. Multiple values of inverse functions.—For any definite value of x there is one and only one value of $\sin x$. This is expressed by saying that $\sin x$ is a **single-valued function** of x. In the examples on finding the values of inverse functions that have been solved in this chapter it has been given that the angles are acute. With this restriction each inverse function is single-valued; but with the restriction removed inverse trigonometric functions have an unlimited number of values.

Thus $\sin^{-1} \frac{1}{2}$ has the value $\frac{\pi}{6}$, or 30°, if the angle is to be acute. Otherwise there is an unlimited number of angles whose sine is $\frac{1}{2}$.

From [12], $\sin \frac{\pi}{6} = \sin \left(\pi - \frac{\pi}{6}\right) = \sin \frac{5\pi}{6}$.

That is,

$$\sin^{-1} \tfrac{1}{2} = \frac{5\pi}{6} \text{ as well as } \frac{\pi}{6}.$$

Again from §129, $\sin (x + 2n\pi) = \sin x$ for all integral values of n (positive or negative).

Hence if $\sin x = \frac{1}{2}$, then x may have any of the values $\frac{\pi}{6} + 2n\pi$ and also any of the values $\frac{5\pi}{6} + 2n\pi$, for all integral values of n. These are all values of $\sin^{-1} \frac{1}{2}$.

In a similar manner it may be shown that each of the other inverse functions has an unlimited number of values.

Hence inverse trigonometric functions are said to be **multiple-valued.** On the following pages we will study the problem of finding all angles which have a given value of one of its functions.

138. The principal values of inverse functions.—The smallest positive value of an inverse function is called its **principal value.**

Thus $\frac{\pi}{6}$, or 30°, is the principal value of $\sin^{-1} \frac{1}{2}$, since this is the smallest positive angle whose sine is $\frac{1}{2}$. Again $\frac{3\pi}{2}$, or 270°, is the principal value of $\sin^{-1}(-1)$, since this is the smallest positive angle whose sine is -1.

EXERCISES

Find the principal value of each of the following, and also two other values of each one:

1. $\sin^{-1} \frac{1}{2}$.
2. $\sin^{-1} (-\frac{1}{2})$.
3. $\sin^{-1} \frac{\sqrt{2}}{2}$.
4. $\sin^{-1} \left(-\frac{\sqrt{2}}{2}\right)$.
5. $\sin^{-1} 1$.
6. $\sin^{-1} (-1)$.
7. $\cos^{-1} \frac{1}{2}$.
8. $\cos^{-1} (-\frac{1}{2})$.
9. $\cos^{-1} \frac{\sqrt{2}}{2}$.
10. $\cos^{-1} \left(-\frac{\sqrt{2}}{2}\right)$.
11. $\cos^{-1} 1$.
12. $\cos^{-1} (-1)$.
13. $\tan^{-1} 1$.
14. $\tan^{-1} (-1)$.
15. $\tan^{-1} \sqrt{3}$.
16. $\tan^{-1} (-\sqrt{3})$.
17. $\tan^{-1} \frac{\sqrt{3}}{3}$.
18. $\tan^{-1} \left(-\frac{\sqrt{3}}{3}\right)$.
19. $\tan^{-1} \frac{\sqrt{2}}{2}$.
20. $\tan^{-1} \left(-\frac{\sqrt{2}}{2}\right)$.
21. $\cot^{-1} 1$.
22. $\cot^{-1} (-1)$.
23. $\cot^{-1} \frac{\sqrt{3}}{3}$.
24. $\cot^{-1} \sqrt{3}$.
25. $\cot^{-1} \left(-\frac{\sqrt{3}}{3}\right)$.
26. $\cot^{-1} (-\sqrt{3})$.
27. $\sec^{-1} 1$.
28. $\sec^{-1} (-1)$.
29. $\sec^{-1} \sqrt{2}$.
30. $\sec^{-1} \sqrt{3}$.
31. $\sec (-\sqrt{2})$.
32. $\csc^{-1} 1$.
33. $\csc^{-1} (-1)$.
34. $\csc^{-1} \sqrt{3}$.
35. $\csc^{-1} \sqrt{5}$.
36. $\csc^{-1} (-\sqrt{3})$.

139. Angles having the same sine.—Let c be a number between -1 and $+1$ (or possibly equal to -1 or to $+1$).

If c is positive there is an angle A in the first quadrant such that $\sin A = c$. If c is negative there is an angle in the third quadrant such that $\sin A = c$.

In either case A is the principal value of $\sin^{-1} c$ and by [12] $\sin A = \sin (\pi - A)$ for all values of A.

By § 129

$$\sin A = \sin (A + 2n\,\pi). \; . \; . \tag{1}$$

and

$$\sin (\pi - A) = \sin (\pi - A + 2n\pi) = \sin [(2n + 1)\,\pi - A], \tag{2}$$

for all integral values of n.

These may be written

$$\sin A = \sin (n\,\pi + A)\ (3) \text{ and } \sin A = \sin (n\,\pi - A) \tag{4}$$

with the provision that n shall be even in (3) and odd in (4).

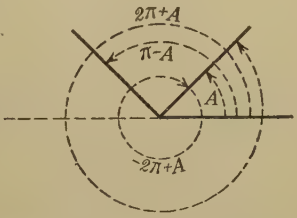
The expression $\sin\,[n\,\pi + (-1)^n A]$ includes both of these, since $(-1)^n = 1$ for even values of n and $(-1)^n = -1$ for odd values of n. Hence we know that $n\,\pi + (-1)^n A$ includes all angles whose sines are equal to $\sin A$.

That is, if $\sin x = \sin A$, then $x = n\,\pi + (-1)^n A$.

Hence,

$$\sin A = \sin\,[n\,\pi + (-1)^n A] \tag*{[62]}$$

Sight Work



1. Find the values of $n\pi + (-1)^n A$ for $A = 60°$ and $n = 0, n = 1, n = 2, n = 3, n = -1, n = -2, n = -3$.

2. Using the figure in the margin, point out the angles A, $\pi - A$, $2\pi + A$, $-2\pi + A$ and show why their sines are equal.

3. Construct a similar figure making $A = 215°$ and show why the general formula gives angles whose sines are equal to sin 215°.

140. Angles having the same cosine.—By a study similar to that just made for the sine we find that

$$\cos A = \cos (2n\,\pi \pm A) \tag*{[63]}$$

for all integral values of n.

That is, if $\cos x = \cos A$, then, $x = 2n\,\pi \pm A$.

141. Angles having the same secant and cosecant.—From the relations

$$\sec A = \frac{1}{\cos A} \text{ and } \csc A = \frac{1}{\sin A}$$

it follows that

$$\sec A = \sec (2\,n\,\pi \pm A) \tag*{[64]}$$

and

$$\csc A = \csc\,[n\,\pi + (-1)^n A] \tag*{[65]}$$

142. Angles having the same tangent and cotangent.—The tangent is positive in the first and third quadrants, and negative in the second and fourth.

Hence, if $\tan x = \tan A$, then x terminates either in the same quadrant as A (is coterminal with A), or it terminates in the opposite quadrant from A. This is a direct consequence of the formula

$$\tan x = \tan (x \pm \pi). \qquad [60]$$

From this formula it also follows that

$$\tan A = \tan (n\,\pi + A) \qquad [66]$$

where n takes all integral values, both positive and negative.

Hence, if $\qquad \tan x = \tan A,$

then, $\qquad x = n\,\pi + A.$

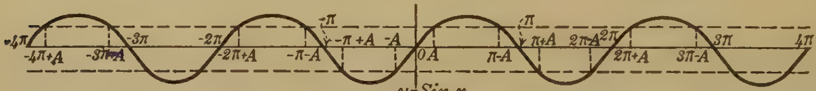
In exactly the same manner we obtain

$$\cot A = \cot (n\,\pi + A). \qquad [67]$$

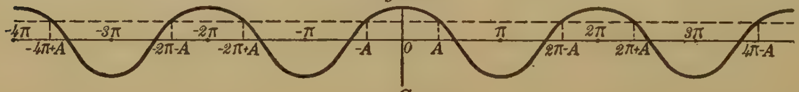
Hence, if $\qquad \cot x = \cot A,$

then, $\qquad x = n\,\pi + A.$

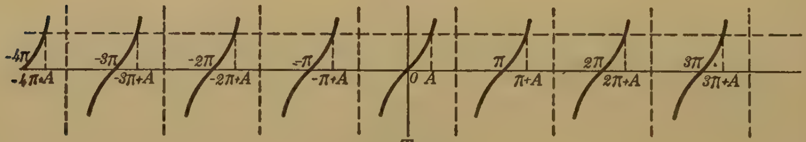
Formulas [62] . . . [67] may be verified readily from the graphs of the trigonometric functions.



y=Sin x



y=Cos x



y=Tan x

EXERCISES

1. From the first graph find a series of angles whose sines are $\frac{3}{4}$, and also a series whose sines are $-\frac{3}{4}$. How would you find from the graph a series whose sines are 4/7? whose sines are 1? whose sines are -1?

2. From the second graph find a series of angles whose cosines are $\frac{1}{2}$. How would you find a series of angles whose cosines are $-\frac{1}{2}$?

3. From the third graph find a series of angles whose tangents are 1. How would you find a series of angles whose tangents are $-5/6$?

143. Identities involving inverse functions.—From the converse statement in §8 it follows that

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

In this equation it is understood that x is positive and that $\sin^{-1} x$ and $\cos^{-1} x$ both represent principal values.

Similarly, under the same restrictions, we have

$$\tan^{-1} y + \cot^{-1} y = \frac{\pi}{2}$$

and

$$\sec^{-1} y + \csc^{-1} y = \frac{\pi}{2}.$$

Example 1. Prove, $\sin(\cos^{-1} u) = \sqrt{1 - u^2}$.

Proof: Let $\cos^{-1} u = A$.

Then,

$$\cos A = u$$

and

$$\sin A = \sqrt{1 - u^2}.$$

That is,

$$\sin(\cos^{-1} u) = \sqrt{1 - u^2}.$$

Example 2. Prove, $\tan u^{-1} + \tan^{-1} v = \tan^{-1} \frac{u + v}{1 - uv}$.

Proof: Let $\tan^{-1} u = A$ and $\tan^{-1} v = B$.

Then,

$$\tan A = u \text{ and } \tan B = v$$

and

$$\tan(A + B) = \frac{u + v}{1 - uv}.$$

That is,

$$A + B = \tan^{-1} \frac{u + v}{1 - uv} \text{ or } \tan^{-1} u + \tan^{-1} v = \tan^{-1} \frac{u + v}{1 - uv}.$$

Example 3. Prove, $\tan^{-1} u + \tan^{-1} v + \tan^{-1} w = \tan^{-1} \frac{u + v + w - uvw}{1 - uv - vw - wu}$.

Proof: Let $\tan^{-1} u = A$, $\tan^{-1} v = B$, $\tan^{-1} w = C$.

Then,

$$\tan A = u,\ \tan B = v,\ \tan C = w,$$

and

$$\tan(A + B + C) = \tan(A + \overline{B + C}) = \frac{\tan A + \tan(B + C)}{1 - \tan A \tan(B + C)}$$

$$= \frac{u + \dfrac{v + w}{1 - vw}}{1 - u\,\dfrac{v + w}{1 - vw}} = \frac{u + v + w - uvw}{1 - uv - vw - wu}.$$

Hence,

$$\tan^{-1} u + \tan^{-1} v + \tan^{-1} w = \tan^{-1} \frac{u + v + w - uvw}{1 - uv - vw - wu}.$$

Sight Work

Find the values of the following:

1. $\sin (\cos^{-1} \frac{1}{2})$.
2. $\cos (\sin^{-1} \frac{3}{4})$.
3. $\tan (\tan^{-1} 1 + \tan^{-1} \frac{1}{2})$.
4. $\sin (\cos^{-1} \frac{1}{2} \sqrt{2})$.
5. $\tan (2 \tan^{-1} x)$.
6. $\cot (2 \cot^{-1} x)$.
7. $\tan (\tan^{-1} x - \tan^{-1} y)$.
8. $\tan (\tan^{-1} 2 - \tan^{-1} 1)$.
9. $\cot (\cot^{-1} 1 + \cot^{-1} 2)$.
10. $\cot (\tan^{-1} \frac{1}{2} \sqrt{3})$.
11. $\tan (\cot^{-1} 3)$.
12. $\tan (\tan^{-1} a + \tan^{-1} 3a)$.

EXERCISES

Prove the following:

1. $\sin (\sin^{-1} x + \sin^{-1} y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$.
2. $\tan (\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}) = 1$.
3. $\tan (\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) = 0$.
4. $\tan^{-1} \dfrac{x}{\sqrt{1 - x^2}} = \sin^{-1} x$.
5. $\tan^{-1} x + \cot^{-1} (x + 1) = \tan^{-1} (x^2 + x + 1)$.
6. $\tan \left[\tan^{-1} \dfrac{a - b}{1 + ab} + \tan^{-1} \dfrac{b - c}{1 + bc} + \tan^{-1} \dfrac{c - a}{1 + ac}\right] = 0$.
7. $\sin (\sin^{-1} x + \cos^{-1} y) = xy + \sqrt{1 - x^2} \cdot \sqrt{1 - y^2}$.
8. $\tan (\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{4}) = 1\frac{2}{3}$.
9. $\tan \left[\tan^{-1} \dfrac{x}{x - 1} - \tan^{-1} \dfrac{x - 1}{x}\right] = \dfrac{2x - 1}{2x(x - 1)}$.
10. $\tan \left[\tan^{-1} \dfrac{x - 1}{x + 2} + \tan^{-1} \dfrac{x + 1}{x + 2}\right] = \dfrac{2x(x + 2)}{4x + 5}$.

Find the values of the following:

11. $\tan (\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4})$.
12. $\tan (\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{8})$.
13. $\tan \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \dfrac{1}{5} + \tan^{-1} \dfrac{1}{7} + \tan^{-1} \dfrac{1}{9}\right)$.
14. $\tan \left[\tan^{-1} \dfrac{x + 1}{x - 1} + \tan^{-1} \dfrac{x - 1}{x}\right]$.
15. $\sin (\sin^{-1} x + \sin^{-1} 2x)$.
16. $\sin (\sin^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{4})$.
17. $\sin \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{4} + \tan^{-1} \dfrac{1}{6}\right)$.
18. $\cos \left(\sin^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{6} + \sin^{-1} \dfrac{1}{12}\right)$.
19. $\tan \left(\cos^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{4} + \cos^{-1} \dfrac{1}{8}\right)$.
20. $\cos \left(\tan^{-1} \dfrac{1}{5} + \tan^{-1} \dfrac{1}{10} + \tan^{-1} \dfrac{1}{20}\right)$.
21. $\sin (\cot^{-1} \frac{1}{2} + \cot^{-1} \frac{1}{3} + \cot^{-1} \frac{1}{4})$.

144. Trigonometric equations.—A number of simple trigonometric equations have already been solved (pages 112–116).

Thus it was found that if $\sin x = \sin A$, then $x = n\pi + (-1)^n A$.

If the equation is given in the form $\sin x = c$ and if $\sin^{-1} c$ is taken to represent only the principal value, then one value of x is $x = \sin^{-1} c$, and all the values of x which satisfy the equation $\sin x = c$ are given by

$$x = n\pi + (-1)^n \sin^{-1} c.$$

We will now consider the relation

$$\sin^2 x - \sqrt{2}\sin x + \tfrac{1}{2} = 0. \tag{1}$$

This equality is certainly not an identity, since it is not satisfied for all values of x. It does not hold, for instance, for $x = 0$.

Solving this equation for $\sin x$ we find

$$\sin x = \tfrac{1}{2}\sqrt{2}. \tag{2}$$

Hence equation (1) is satisfied by $x = \frac{\pi}{4}$.

By [62], the set of all values of x which satisfy (2) is given by the expression

$$x = n\pi + (-1)^n \frac{\pi}{4}.$$

Therefore this expression gives all values of x which satisfy (1).

145. Solving trigonometric equations.—Solving a trigonometric equation consists in finding all values of the unknown angle (or angles) which satisfy the equation.

Example 1. Solve $4 \sin x = \csc x$.

Solution: We have

$$4\sin x = \frac{1}{\sin x} \quad \text{or } 4\sin^2 x = 1.$$

Hence,

$$\sin x = \pm \tfrac{1}{2},$$

and by [62],

$$x = n\pi + (-1)^n \frac{\pi}{6}, \text{ and } x = n\pi - (-1)^n \frac{\pi}{6}.$$

Example 2. Solve $2\cos^2 x + 7\sin x - 5 = 0$.

Solution: We have

$$2(1 - \sin^2 x) + 7\sin x - 5 = 0,$$

$$2\sin^2 x - 7\sin x + 3 = 0,$$

$$\sin x = \frac{7 \pm \sqrt{49 - 24}}{4} = 3, \tfrac{1}{2}.$$

Since 3 cannot be the value of the sine of an angle, it follows that $\frac{1}{2}$ is the only value of the sine which satisfies this equation.

Hence by [62],

$$x = n\pi + (-1)^n \frac{\pi}{6}.$$

Example 3. Solve, $\tan x - 6 \cot x + 1 = 0$.

Solution: We have

$$\tan x - \frac{6}{\tan x} + 1 = 0,$$

$$\tan^2 x - 6 + \tan x = 0.$$

Hence

$$\tan x = 2 \text{ or } -3,$$

and by [66],

$$x = n\pi + \tan^{-1} 2, \quad x = n\pi + \tan^{-1}(-3),$$

$\tan^{-1} 2$ and $\tan^{-1}(-3)$ being understood to represent principal values only.

Example 4. Solve, $\cos 2x \sec x + \sec x + 1 = 0$.

Solution: We have

$$\frac{(\cos^2 x - \sin^2 x)}{\cos x} + \frac{1}{\cos x} + 1 = 0,$$

$$\cos^2 x - \sin^2 x + 1 + \cos x = 0,$$

$$2\cos^2 x + \cos x = 0,$$

$$\cos x(2\cos x + 1) = 0.$$

Hence

$$\cos x = 0 \text{ and } \cos x = -\tfrac{1}{2}.$$

But $\cos x = 0$ does not satisfy the given equation, since this makes $x = \frac{\pi}{2}$ (or $\frac{3n}{2}\pi$), and for this value of x the secant is not defined. From $\cos x = -\frac{1}{2}$, $x = 2n\pi \pm \frac{2\pi}{3}$.

No general rule can be given for solving trigonometric equations. It is usually convenient, however, first to reduce the equation so it will contain only one function of the same angle. In this respect note all the examples given above.

EXERCISES

Solve the following equations:

1. $\cos x = \cot^2 x$.
2. $\cos 2x = 2 \sin x$.
3. $\sin 2x = 2 \cos x$.
4. $\sin x = \cos 2x$.
5. $\sec x = 2 \csc x$.
6. $\cos 2x = -\cos x$.
7. $2 \sin^2 x + 3 \cos x = 0$.
8. $\sin x + \cos x = \sqrt{2}$.
9. $\cos 2x + \cos x = -1$.
10. $\cos 2x = \cos^2 x$.
11. $\sin^2 x - 2 \cos x + \frac{1}{4} = 0$.
12. $\cos 2x + \sin x = 1$.
13. $\tan 2x + 2 \sin x = 0$.
14. $3 \cos x + 5 \sin x = 4$.
15. $\sin x + 2 \cos x = 1$.
16. $\sec 2x = 2 \cos x - 1$.
17. $2 \sin^2 x + \sin^2 2x = 2$.
18. $\sin x \sec 2x = 1$.
19. $\tan 2x \tan x = 1$.
20. $\sin^2 x - \cos^2 x = 0$.
21. $\cos x \cos 3x = -\frac{1}{2}$.
22. $\cos 2x = \sin 4x$.
23. $\tan^2 x + \cot^2 x = 2$.
24. $\sec^2 x + 3 \tan^2 x = 5$.
25. $\cos 3x + \sin x = \cos x$.
26. $\sin x + \sin 2x + \sin 3x = 0$.

146. Simultaneous trigonometric equations.—The examples given below illustrate the more important types of simultaneous trigonometric equations.

Example 1. Solve for x and y the equations:

$$\begin{cases} \sin x + \sin y = a & (1) \\ \cos x + \cos y = b & (2) \end{cases}$$

Solution: By [32],

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y) = a, \tag{3}$$

and by [34],

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y) = b. \tag{4}$$

Dividing (3) by (4),

$$\tan \tfrac{1}{2}(x + y) = \frac{a}{b}. \tag{5}$$

Therefore,

$$\sin \tfrac{1}{2}(x + y) = \pm \frac{a}{\sqrt{a^2 + b^2}}. \tag{6}$$

Substituting (6) in (3),

$$\cos \tfrac{1}{2}(x - y) = \pm \tfrac{1}{2}\sqrt{a^2 + b^2}. \tag{7}$$

From (5),

$$x + y = 2 \tan^{-1} \frac{a}{b}.$$

From (7),

$$x - y = 2 \cos^{-1}\left(\pm \tfrac{1}{2}\sqrt{a^2 + b^2}\right).$$

Hence,

$$x = \tan^{-1} \frac{a}{b} + \cos^{-1}\left(\pm \tfrac{1}{2}\sqrt{a^2 + b^2}\right),$$

$$y = \tan^{-1} \frac{a}{b} - \cos^{-1}\left(\pm \tfrac{1}{2}\sqrt{a^2 + b^2}\right).$$

The general solution is:

$$x = 2n\pi + \tan^{-1} \frac{a}{b} + \cos^{-1}\left(\pm \tfrac{1}{2}\sqrt{a^2 + b^2}\right),$$

$$y = 2n\pi + \tan^{-1} \frac{a}{b} - \cos^{-1}\left(\pm \tfrac{1}{2}\sqrt{a^2 + b^2}\right)$$

where the inverse functions represent principal values.

Thus, if it is given that

$$\begin{cases} \sin x + \sin y = \tfrac{1}{2}, \\ \cos x + \cos y = \dfrac{3}{2}, \end{cases}$$

then,

$$x = 2n\pi + \tan^{-1} \frac{1}{3} + \cos^{-1}\left(\pm \frac{1}{4}\sqrt{10}\right),$$

$$y = 2n\pi + \tan^{-1} \frac{1}{3} - \cos^{-1}\left(\pm \frac{1}{4}\sqrt{10}\right).$$

Example 2. Solve for r and x the system

$$\begin{cases} r \sin x = a, & (1) \\ r \cos x = b. & (2) \end{cases}$$

Solution: Squaring both members of (1) and (2) and adding,

$$r^2 (\sin^2 x + \cos^2 x) = a^2 + b^2, \text{ or } r = \pm \sqrt{a^2 + b^2}.$$

Dividing (1) by (2),

$$\tan x = \frac{a}{b},$$

and

$$x = \tan^{-1} \frac{a}{b}.$$

The sign of $\sqrt{a^2 + b^2}$ must be so chosen that (1) and (2) will be satisfied. Thus, if

$$r \sin x = 4,$$
$$r \cos x = 6,$$

Then,

$$r = \pm 2\sqrt{13} \qquad \text{and } x = \tan^{-1} \frac{2}{3}.$$

If the principal value of $\tan^{-1} \frac{2}{3}$ is taken, r must be positive. That is, in this case $r = 2\sqrt{13}$.

Example 3. Solve for r, x, y the system

$$\begin{cases} r \sin x \cos y = a, & (1) \\ r \sin x \sin y = b, & (2) \\ r \cos x = c. & (3) \end{cases}$$

Solution: Dividing (2) by (1),

$$\tan y = \frac{b}{a} \text{ and } y = \tan^{-1} \frac{b}{a}.$$

Squaring (1), (2), (3) and adding

$$\begin{aligned} r^2 (\sin^2 x \cos^2 y + \sin^2 x \sin^2 y + \cos^2 x) &= a^2 + b^2 + c^2 \\ &= r^2 [\sin^2 x (\cos^2 y + \sin^2 y) + \cos^2 x] \\ &= r^2 (\sin^2 x + \cos^2 x) = r^2. \end{aligned}$$

Hence,

$$r^2 = a^2 + b^2 + c^2 \quad \text{or} \quad r = \pm \sqrt{a^2 + b^2 + c^2}.$$

From (3)

$$\cos x = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

or

$$x = \cos^{-1}\left(\pm \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right).$$

Hence the complete solution is

$$r = \pm\sqrt{a^2+b^2+c^2}$$

$$x = \cos^{-1}\left(\pm\frac{c}{\sqrt{a^2+b^2+c^2}}\right) + 2n\pi$$

$$y = \tan^{-1}\frac{b}{a} + 2n\pi.$$

The signs of $\frac{1}{2}\sqrt{a^2+b^2}$ and of $\sqrt{a^2+b^2+c^2}$ in Exs. 1 and 3 must be taken so that original equations will be satisfied.

EXERCISES

Solve the following systems of equations:

1. $\begin{cases}\sin x + \sin y = \frac{1}{3}\\ \cos x + \cos y = \frac{5}{4}.\end{cases}$

2. $\begin{cases}\sin x - \sin y = \frac{1}{4}\\ \cos x - \cos y = \frac{1}{3}.\end{cases}$

3. $\begin{cases}\cos x + \cos y = a\\ \cos 2x + \cos 2y = b.\end{cases}$

4. $\begin{cases}r \sin x \cos y = 2\\ r \sin x \sin y = 3\\ r \cos y = 4.\end{cases}$

5. $\begin{cases}r \sin x \cos y = 8\\ r \sin x \sin y = 10\\ r \cos y = 12.\end{cases}$

6. $\begin{cases}r \sin x = 4\\ r \cos x = 2.\end{cases}$

7. $\begin{cases}r \sin x = 2\sqrt{5}\\ r \cos x = -3\sqrt{3}.\end{cases}$

8. $\begin{cases}r \sin x = -3\sqrt{2}\\ r \cos x = -4\sqrt{5}.\end{cases}$

9. $\begin{cases}r \sin x \cos y = \sqrt{5}\\ r \sin x \sin y = \sqrt{7}\\ r \cos y = \sqrt{10}.\end{cases}$

10. $\begin{cases}r \sin x \cos y = 2\\ r \sin x \sin y = \sqrt{5}\\ r \cos y = \sqrt{7}.\end{cases}$

147. Equations stated in terms of inverse functions.—An equality of the form

$$\cos^{-1} x + \cos^{-1} 2x = \cos^{-1}\tfrac{1}{2}$$

is not an identity and hence presents a problem for solution.

Example 1. Solve for x, $\cos^{-1} x + \cos^{-1} 2x = \cos^{-1}\frac{1}{2}$.

Solution: Let $\cos^{-1} x = A$ and $\cos^{-1} 2x = B$.

Then,

$$\cos A = x \text{ and } \cos B = 2x,$$

Hence,

$$\sin A = \sqrt{1-x^2},\ \sin B = \sqrt{1-4x^2}$$

and

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B\\ &= x \cdot 2x - \sqrt{1-x^2}\cdot\sqrt{1-4x^2}.\end{aligned}$$

That is,

$$2x^2 - \sqrt{1-x^2}\cdot\sqrt{1-4x^2} = \tfrac{1}{2}.$$

Then,

$$(2x^2 - \tfrac{1}{2})^2 = (1 - x^2)(1 - 4x^2)$$

or

$$4x^4 - 2x^2 + \tfrac{1}{4} = 1 - 5x^2 + 4x^4,$$

or

$$3x^2 = \tfrac{3}{4} \text{ and } x = \pm \tfrac{1}{2}.$$

If $\cos A = \frac{1}{2}$, then $A = \frac{\pi}{3}$ and if $\cos B = 1$, then $B = 0$.

Since in this case $A + B = \frac{\pi}{3}$, and since $\cos \frac{\pi}{3} = \frac{1}{2}$, the given equation is satisfied.

If $\cos A = -\frac{1}{2}$, then $A = \frac{2\pi}{3}$, and if $\cos B = -1$, then $B = \pi$, and $A + B = \pi + \frac{2\pi}{3}$.

Since $\cos\left(\pi + \frac{2\pi}{3}\right) = \cos \frac{5\pi}{3} = \frac{1}{2}$, the given equation is satisfied.

Hence $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ both satisfy the given equation.

Example 2. $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1} \frac{1}{4}$.

Solution: Let $\tan^{-1}(x + 1) = A$ and $\tan^{-1}(x - 1) = B$.

Then,

$$\tan A = x + 1 \text{ and } \tan B = x - 1$$

and

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x + 1 + x - 1}{1 - (x + 1)(x - 1)}.$$
$$= \frac{2x}{2 - x^2}$$

Hence,

$$\frac{2x}{2 - x^2} = \tfrac{1}{4} \text{ and } x = -4 \pm 3\sqrt{2}.$$

EXERCISES

Solve the following equations:

1. $\sin^{-1} x = \cos^{-1} x$.
2. $2 \sin^{-1} x = 3 \cos^{-1} x$.
3. $\tan^{-1} x = \cot^{-1} x$.
4. $\sin^{-1} \frac{5}{x} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}$.
5. $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$.
6. $\tan^{-1} x + 2 \cot^{-1} x = \frac{3\pi}{4}$.
7. $\sin^{-1} x + 3 \cos^{-1} x = \frac{7\pi}{6}$.
8. $\cot^{-1} x + \cot^{-1} 2x = \frac{3\pi}{4}$.
9. $\cos^{-1} x + \cos^{-1} \sqrt{1 - x^2} = \cos^{-1} x\sqrt{3}$.
10. $\tan^{-1} \frac{x - 1}{x + 2} + \tan^{-1} \frac{x + 1}{x + 2} = \frac{\pi}{4}$.
11. $2 \tan^{-1} \frac{1}{2} + \sin^{-1} \frac{4}{5} = \sin^{-1} \frac{1}{x}$.
12. $\tan^{-1} \frac{x + 1}{x - 1} + \tan^{-1} \frac{x - 1}{x} = \tan^{-1}(-7)$.

CHAPTER XI

DE MOIVRE'S THEOREM

THIS chapter contains the beginnings of a theory which uses trigonometric functions in the study of complex numbers. While at the outset this theory appears to be purely abstract and of no practical value, it eventually leads to important applications in various branches of mathematics and also in physics, mechanics, and in practical engineering.

148. Real numbers.—The ordinary numbers of arithmetic indicate magnitude only, while the *real number system* of algebra includes both positive and negative numbers, and hence the numbers of algebra represent both magnitude and quality.

The numerals $+1$ and -1 may be regarded as quality units, and any real number may be regarded as a multiple of one of these. Thus $+5$ may be regarded as $5(+1)$ and -5 as $5(-1)$. The multiplier 5 is a number which represents magnitude only, and is the kind of number which is used in elementary arithmetic. A real number therefore is the product of an arithmetic number and a quality unit. The arithmetic multiplier is called the **arithmetic value** or the **absolute value** of the real number.

149. Imaginary numbers.—Since the square of any real number is positive whether the number itself is positive or negative, it follows that in the real number system there is no square root of a negative number. Thus there is no real number which is the root of an equation like $x^2 = -1$, $x^2 = -2$, or $x^2 = -25$.

Imaginary numbers were deliberately invented so that every algebraic equation might have a root. The essential element of this system is the **imaginary unit,** $\sqrt{-1}$, also designated by i, which has the property that $i^2 = -1$, or $(\sqrt{-1})^2 = -1$.

An imaginary number is a product of a real number and the imaginary unit. Thus $6i$, $-3i$, etc., are imaginary numbers.

The square of an imaginary number is always a negative real number. Thus $(6i)^2 = 6^2i^2 = 36\ (-1) = -36$,

and $(-3i)^2 = (-3)^2\ (i)^2 = 9\,(-1) = -9$.

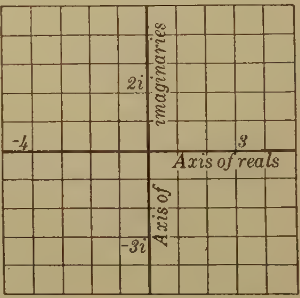
Both imaginary and real numbers may be considered as special cases of the complex number which we now proceed to study.

150. Complex numbers.—A number expressed in the form $a + bi$, where a and b are real numbers, is a **complex number.** Thus $2 + 3i$, $\sqrt{2} - 2i$, and $-6 + \sqrt{3}i$ are complex numbers.

The complex number $a + bi$ is thus seen to consist of the sum of a real number a and an imaginary number bi (or ib). The a is called the *real* part of the number and bi the imaginary part.

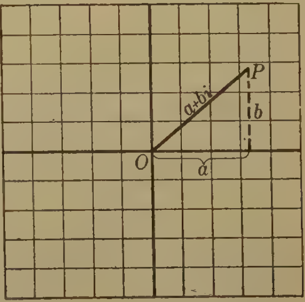
151. Graphical representation of complex numbers.—Complex numbers admit of a convenient geometric representation by what is known as the Argand diagram. Use is made of ordinary square-ruled paper with a pair of mutually perpendicular axes. Real numbers are represented by line-segments extending from the origin to points on the horizontal axis, or **axis of reals.** As in the figures studied earlier in this course (see page 58 *et seq.*), distances to the right of the origin represent positive numbers and distances to the left represent negative numbers.

Imaginary numbers are represented on the vertical axis, or the **axis of imaginaries.** Thus $2i$ is represented by a point on this axis two units above the origin, and $-3i$ is represented by a point three units below the origin.



Since the origin lies on both axes, the number zero belongs to the system of real numbers and also to the system of imaginary numbers, and is the only number that has this property.

To represent a complex number $a + bi$ graphically, lay off a along the axis of reals beginning at the origin as described above, and from the extremity of this segment lay off b in the vertical direction, reaching the point P in the figure. The line-segment from the origin to P is the graphic representation of the complex number $a + bi$. The point P itself may also be regarded as a representation of this number.



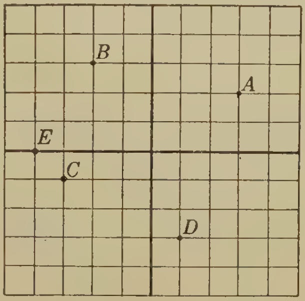
In this manner every point in the plane is made to represent a complex number, and conversely, every complex number is represented by such a point

It should be noted that in $a + bi$, the real numbers a and b may have any values whatever including zero. It follows that real numbers and also pure imaginaries are special cases of the complex number. If $b = 0$, $a + bi$ reduces to $a + 0 \cdot i = a$, and the corresponding point lies on the axis of reals. If $a = 0$, $a + bi = 0 + bi = bi$, which is a pure imaginary, and the corresponding point lies on the axis of imaginaries. If $a = 0$ and $b = 0$, the corresponding point is in the origin.

SIGHT WORK

1. What numbers are represented by the points A, B, C, D, E, in the figure?

2. On a graph mark the points corresponding to $1 + 2i$, $3 - 2i$, $-2 - 2i$, $-2 + 2i$, $2 - i$.

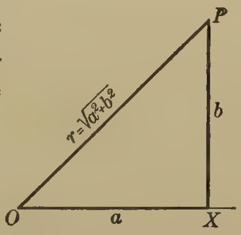


152. Equality of complex numbers.—Two complex numbers are equal only when their real parts are equal and also their imaginary parts.

That is, $a + bi = c + di$ if, and only if, $a = c$ and $b = d$.

This is seen at once from the graphic representation of complex numbers.

153. Type-forms of the complex number.—A complex number may also be regarded as an arithmetic multiple of a *complex unit.* From the graphic representation the line segment which represents the complex number $a + bi$ is seen to be $\sqrt{a^2 + b^2}$ in length.



If we denote this quantity by r we have

$$a + bi = r\left(\frac{a}{r} + \frac{b}{r}i\right)$$

In this expression r is called the **modulus** and $\frac{a}{r} + i\frac{b}{r}$ the **complex unit.** This is one of the type forms of the complex unit and $r\left(\frac{a}{r} + \frac{b}{r}i\right)$ is one of the type forms of the general complex number.

Denoting by φ the angle XOP, we have

$$\frac{a}{r} = \cos\varphi \text{ and } \frac{b}{r} = \sin\varphi.$$

Substituting

$\frac{a}{r} = \cos\varphi$ and $\frac{b}{r} = \sin\varphi$ in its type forms on page 127 gives

$$\mathbf{\cos\varphi + i\sin\varphi \text{ and } r(\cos\varphi + i\sin\varphi)}$$

which are the respective trigonometric type forms of the complex unit and of the general complex number. The angle φ is called the **amplitude** of the complex number.

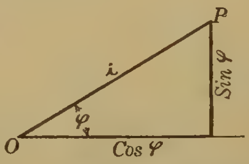
Notice that a line segment representing a complex unit is always of unit length, and that hence a point P representing a complex unit lies on a circle with radius unity and center at the origin.

Since coterminal angles have equal functions it follows that

$$\cos\varphi + i\sin\varphi = \cos(2n\pi + \varphi) + i\sin(2n\pi + \varphi).$$

This latter form may be designated as the general trigonometric type form of the complex unit.

It should be noted that while the modulus of the complex unit is always equal to unity, the amplitude may be any angle whatever. That is, complex units are not equal in the ordinary sense.



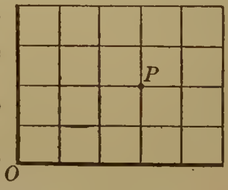
SIGHT WORK

1. Show why the modulus of the number represented by P in the figure is $\sqrt{13}$.

2. Find the sine and the cosine of the amplitude of the number represented by the point marked P in the figure.

3. Find the modulus of the number $3 + 4i$. Also find the sine and cosine of its amplitude.

4. Find the modulus of each of the following numbers: $1 - 7i$, $-5 + 4i$, $5 - 4i$, $-2 - 7i$, $3 - 5i$, $-7 + 3i$.



5. Find the sine and cosine of the amplitude of each of the complex numbers in Example 4.

154. Arithmetic operations on complex numbers.—Two complex numbers such as $a + bi$ and $c + di$ are added by adding the real parts and also the complex parts.

Thus,

$$(a + bi) + (c + di) = a + c + (b + d)i.$$

The product of two complex numbers is found exactly like the product of two ordinary binomials, remembering that $(i)^2 = -1$. Thus,

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bd(i)^2\\ &= ac - bd + (ad+bc)i.\end{aligned}$$

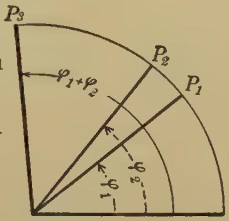
Sight Work

Find the products of **1.** $(3+2i)(1-2i)$, **2.** $(1-3i)(2-i)$, **3.** $(2-7i)(-1+i)$, **4.** $(-4+2i)(3-2i)$, **5.** $(-1+i)(1-i)$, **6.** $(2-i)(2+i)$, **7.** $(-4+7i)(-4-7i)$.

155. Multiplication of complex units.—We shall now consider more in detail the products of complex numbers, and in particular of complex units.

Putting two complex units into the trigonometric type form and multiplying as above, we have:

$$\begin{aligned}&(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2)\\ &= \cos\varphi_1\cdot\cos\varphi_2 + i^2\sin\varphi_1\sin\varphi_2 + i\sin\varphi_1\\ &\quad \cos\varphi_2 + i\sin\varphi_2\cos\varphi_1\\ &= \cos\varphi_1\cdot\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i\,(\sin\varphi_1\\ &\quad \cos\varphi_2 + \sin\varphi_2\cos\varphi_1)\\ &= \cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2).\end{aligned}$$



Thus, it is seen that the product of two complex units is another complex unit whose amplitude is the sum of the amplitudes of the given units. In the figure these three numbers are represented by P_1, P_2, and P_3.

It follows that the product of three complex units $\cos\varphi_1 + i\sin\varphi_1$, $\cos\varphi_2 + i\sin\varphi_2$, $\cos\varphi_3 + i\sin\varphi_3$ is

$$\cos(\varphi_1+\varphi_2+\varphi_3) + i\sin(\varphi_1+\varphi_2+\varphi_3)$$

and that the product of n complex units $\cos\varphi_1 + i\sin\varphi_1, \ldots,$ $\cos\varphi_n + i\sin\varphi_n$, is $\cos(\varphi_1+\varphi_2+\ldots+\varphi_n) + i\sin(\varphi_1+\varphi_2+\ldots+\varphi_n)$.

Sight Work

1. If it is given that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\sin 45° = \frac{1}{2}\sqrt{2}$, $\cos 45° = \frac{1}{2}\sqrt{2}$, find sin 75° and cos 75°.

Suggestion: $(\cos 30° + i\sin 30°)(\cos 45° + i\sin 45°) = \cos 75° + i\sin 75°$. Substitute the given values in the first member and multiply.

2. If it is given that $\sin(-30°) = -\frac{1}{2}$ and $\cos(-30°) = \frac{1}{2}\sqrt{3}$, find $\sin(-60°)$ and $\cos(-60°)$.

3. If it is given that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\sin 60° = \frac{1}{2}\sqrt{3}$, and $\cos 60° = \frac{1}{2}$, show by the method suggested above that $\sin 90° = 1$ and $\cos 90° = 0$.

156. De Moivre's theorem.—If a complex unit is multiplied by itself by the method of §155, we find that a complex unit may be squared by simply doubling its amplitude.

That is,

$$(\cos\varphi + i\sin\varphi)^2 = (\cos\varphi + i\sin\varphi)(\cos\varphi + i\sin\varphi)$$
$$= \cos(\varphi+\varphi) + i\sin(\varphi+\varphi) = \cos 2\varphi + i\sin 2\varphi.$$

Similarly, a complex unit may be cubed by multiplying its amplitude by 3, may be raised to the fourth power by multiplying its amplitude by 4, and so on.

These are special cases of the theorem known as De Moivre's theorem:

The nth power[1] *of a complex unit is a complex unit whose amplitude is n times the amplitude of the base.*

In symbols,

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi \qquad [68]$$

Proof: Case I. Let n be a positive integer k.

Then

$$(\cos\varphi + i\sin\varphi)^k = (\cos\varphi + i\sin\varphi)(\cos\varphi + i\sin\varphi)\ \ldots \text{ to } k \text{ factors.}$$
$$= \cos(\varphi + \varphi + \text{to } k \text{ terms}) + i\sin(\varphi + \varphi + \text{to } k \text{ terms})$$
$$= \cos k\varphi + i\sin k\varphi.$$

Case II. Let n be the fraction $\frac{r}{s}$, r and s being positive integers, and $\frac{r}{s}$ being in its lowest terms.

By Case I it follows that

$$\left(\cos\frac{\varphi}{s} + i\sin\frac{\varphi}{s}\right)^s = \cos\varphi + i\sin\varphi.$$

Taking the sth root of each side,

$$\cos\frac{\varphi}{s} + i\sin\frac{\varphi}{s} = (\cos\varphi + i\sin\varphi)^{\frac{1}{s}}$$

Raising each member to the rth power,

$$\cos\frac{r}{s}\varphi + i\sin\frac{r}{s}\varphi = (\cos\varphi + i\sin\varphi)^{\frac{r}{s}}.$$

Case III Let n be any negative number $-p$. By actual multiplication

$$[\cos\varphi + i\sin\varphi][\cos(-\varphi) + i\sin(-\varphi)] = \cos 0 + i\sin 0 = 1.$$

Hence,

$$\cos(-\varphi) + i\sin(-\varphi) = \frac{1}{\cos\varphi + i\sin\varphi} = (\cos\varphi + i\sin\varphi)^{-1}.$$

That is,

$$(\cos\varphi + i\sin\varphi)^{-1} = \cos(-\varphi) + i\sin(-\varphi).$$

---

[1] The word "power" is used here to denote the result of affecting the number by an exponent n, whether n is integral or fractional, positive or negative.

Raising both members to the power p, by Cases I and II,

$$\begin{aligned}(\cos\varphi + i\sin\varphi)^{-p} &= [\cos(-\varphi) + i\sin(-\varphi)]^p.\\ &= \cos(-p\,\varphi) + i\sin(-p\,\varphi).\end{aligned}$$

This proves the theorem for any rational exponents. It may be extended to the case of irrational exponents by an argument involving the method of limits.

157. Roots of complex units.—Since the square of a complex unit expressed in the form $\cos\varphi + i\sin\varphi$ may be found by doubling its amplitude, it follows that the square root of such a unit may be found by dividing the amplitude by 2.

That is, from $(\cos\varphi + i\sin\varphi)^2 = \cos 2\varphi + i\sin 2\varphi$

it follows that $\sqrt{\cos\varphi + i\sin\varphi} = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}.$

Similarly, it follows that

$$\sqrt[3]{\cos\varphi + i\sin\varphi} = \cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3}$$

$$\sqrt[4]{\cos\varphi + i\sin\varphi} = \cos\frac{\varphi}{4} + i\sin\frac{\varphi}{4}$$

and in general

$$\sqrt[n]{\cos\varphi + i\sin\varphi} = \cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}.$$

We have seen (§153) that

$$\cos\varphi + i\sin\varphi = \cos(2k\,\pi + \varphi) + i\sin(2k\,\pi + \varphi)$$

for all integral values of k.

Hence we have,

$$\sqrt[n]{\cos\varphi + i\sin\varphi} = \cos\frac{2k\,\pi + \varphi}{n} + i\sin\frac{2k\,\pi + \varphi}{n}$$

for all integral values of k.

This means, for instance, that each of the expressions

$$\cos\frac{\varphi}{3} + i\sin\frac{\varphi}{3},\ \cos\frac{2\pi + \varphi}{3} + i\sin\frac{2\pi + \varphi}{3},$$
$$\cos\frac{4\pi + \varphi}{3} + i\sin\frac{4\pi + \varphi}{3},\ \cos\frac{6\pi + \varphi}{3} + i\sin\frac{6\pi + \varphi}{3},\ \ldots$$

is a cube root of $\cos\varphi + i\sin\varphi$.

However, this process gives only three distinct cube roots of a complex unit, since

$$\cos\frac{6\pi+\varphi}{3}+i\sin\frac{6\pi+\varphi}{3}=\cos\frac{\varphi}{3}+i\sin\frac{\varphi}{3},$$

$$\cos\frac{8\pi+\varphi}{3}+i\sin\frac{8\pi+\varphi}{3}=\cos\frac{2\pi+\varphi}{3}+i\sin\frac{2\pi+\varphi}{3},$$

$$\cos\frac{10\pi+\varphi}{3}+i\sin\frac{10\pi+\varphi}{3}=\cos\frac{4\pi+\varphi}{3}+i\sin\frac{4\pi+\varphi}{3}, \text{ etc.}$$

Hence,

$$\cos\frac{\varphi}{3}+i\sin\frac{\varphi}{3},\ \cos\frac{2\pi+\varphi}{3}+i\sin\frac{2\pi+\varphi}{3},$$

$\cos\frac{4\pi+\varphi}{3}+i\sin\frac{4\pi+\varphi}{3}$ are the cube roots of $\cos\varphi+i\sin\varphi$ obtained by this process.

In the same manner we find four fourth roots of $\cos\varphi+i\sin\varphi$, and, if n is a positive integer, n nth roots of $\cos\varphi+i\sin\varphi$.

EXERCISES

1. Show fully how to find the four fourth roots of $\cos\varphi+i\sin\varphi$, and show why there are not more than four distinct fourth roots.

2. In a similar manner show that $\cos\varphi+i\sin\varphi$ has exactly five distinct fifth roots and exactly six sixth roots.

3. Find the eight distinct eighth roots of unity.

158. The roots of any number.—Since any complex number may be expressed in the form $r(\cos\varphi+i\sin\varphi)$, where r and φ are positive real numbers, and since we have seen that there are n nth roots of $\cos\varphi+i\sin\varphi$, and since the positive real number r has one arithmetic nth root, it follows that $r(\cos\varphi+i\sin\varphi)$ has n distinct nth roots. That is,

$$\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=r^{\frac{1}{n}}\left(\cos\frac{2k\pi+\varphi}{n}+i\sin\frac{2k+\varphi}{n}\right) \qquad [69]$$

for all integral values of k. As shown in §157 this gives rise to exactly n distinct roots.

But any number, real or complex, may be expressed in the form $r(\cos\varphi+i\sin\varphi)$.

Thus,

$$2=2(\cos 0+i\sin 0),\ 3i=3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

and

$$3+5i=\sqrt{34}\left[\cos\left(\tan^{-1}\frac{5}{3}\right)+i\sin\left(\tan^{-1}\frac{5}{3}\right)\right].$$

if $\varphi = \tan^{-1}\frac{5}{3}$, then $3 + 5i = \sqrt{34}\,(\cos\varphi + i\sin\varphi)$.

Hence it follows that every number (real, purely imaginary, or complex) has n distinct nth roots.

Example. Find the six sixth roots of $1 + 3\,i$.

Solution: $r = \sqrt{1+9} = \sqrt{10}$, and $\varphi = \tan^{-1} 3 = 71°\,33'\,54''$. Hence,

$$\sqrt[6]{1+3\,i} = (\sqrt{10})^{\frac{1}{6}}\left(\cos\frac{71°\,33'\,54''}{6} + i\sin\frac{71°\,33'\,54''}{6}\right)$$

$$= (10)^{\frac{1}{12}}\,(\cos 11°\,55'\,39'' + i\sin 11°\,55'\,39'')$$

$$= 10^{\frac{1}{12}}\,[.9784 + i\,(.2066)],\text{ is one of the sixth roots of } 1 + 3i.$$

From

$$10^{\frac{1}{12}}\left(\cos\frac{360° + 71°\,33'\,54''}{6} + i\sin\frac{360° + 71°\,33'\,54''}{6}\right)$$

we obtain another sixth root, and so on. If it is desired to find a decimal approximation of $(10)^{\frac{1}{12}}$, logarithms should be used.

Hence, the n nth roots of a number, n being a positive integer, may be obtained through the following steps:

(a) *Put the number into the general trigonometric type form of a complex number.*

(b) *Apply De Moivre's theorem.*

(c) *Assign to k the values* 0, 1, 2, . . ., $n - 1$.

EXERCISES

1. Find the five fifth roots of $1 + i$.

Suggestion:

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Hence,

$$\sqrt[5]{1+i} = 2^{\frac{1}{10}}\left(\cos\frac{2k\pi + \frac{\pi}{4}}{5} + i\sin\frac{2k\pi + \frac{\pi}{4}}{5}\right).$$

Assign the values 0, 1, 2, 3, 4 to k and find the roots.

2. Find the three cube roots of unity and compare with the roots of $x^3 - 1 = 0$.

3. Find the three cube roots of i and check by raising each to the third power.

4. Find the four fourth roots of i and check by raising each to the fourth power.

159. Formulas for sine and cosine of multiple angles.—By using De Moivre's theorem, formulas for the sine and cosine of multiple angles may be obtained as follows:

$$(\cos \varphi + i \sin \varphi)^n = \cos n \varphi + i \sin n \varphi.$$

Expanding the left member by the Binomial Theorem, we have

$$\cos n \varphi + i \sin n \varphi = \cos^n \varphi + n \cos^{n-1} \varphi (i \sin \varphi) + \frac{n(n-1)}{2} \cos^{n-2} \varphi \, (i \sin \varphi)^2 + \frac{n(n-1)(n-2)}{2 \cdot 3} \cos^{n-3} \varphi \, (i \sin \varphi)^3 + \ldots.$$

Noting that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc., and collecting real and imaginary terms, we have

$$\cos n \varphi + i \sin n \varphi = \cos^n \varphi - \frac{n(n-1)}{2} \cos^{n-2} \varphi \sin^2 \varphi + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} \cos^{n-4} \varphi \sin^4 \varphi \ldots + i \left[n \cos^{n-1} \varphi \sin \varphi - \frac{n(n-1)(n-2)}{2 \cdot 3} \cos^{n-3} \varphi \sin^3 \varphi + \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4 \cdot 5} \cos^{n-5} \varphi \sin^5 \varphi \ldots \right]$$

Then by §152,

$$\mathbf{\cos n \varphi = \cos^n \varphi - \frac{n(n-1)}{2} \cos^{n-2} \varphi \sin^2 \varphi + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} \cos^{n-4} \varphi \sin^4 \varphi - \ldots} \qquad [70]$$

$$\mathbf{\sin n \varphi = n \cos^{n-1} \varphi \sin \varphi - \frac{n(n-1)(n-2)}{2 \cdot 3} \cos^{n-3} \varphi \sin^3 \varphi + \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4 \cdot 5} \cos^{n-5} \varphi \sin^5 \varphi - \ldots} \qquad [71]$$

If $n = 2$, then by [70]

$$\cos 2 \varphi = \cos^2 \varphi - \frac{2(2-1)}{2} \cos^{2-2} \varphi \sin^2 \varphi = \cos^2 \varphi - \sin^2 \varphi$$

and by [71],

$$\sin 2 \varphi = 2 \cos^{2-1} \varphi \sin \varphi = 2 \cos \varphi \sin \varphi.$$

Notice that in the formulas for $\cos n \varphi$ and $\sin n \varphi$ the third and second terms, respectively, and all terms that follow them, contain the factor $n - 2$, which equals zero when $n = 2$. Hence all these terms are zero for $n = 2$. Similarly, for $n = 3$ all terms after the third and second, respectively, disappear.

CHAPTER XII

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

In the calculus the following series are developed.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad \text{(A)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad \text{(B)}$$

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \text{(C)}$$

These are known as the sine, cosine, and exponential series, respectively. In series (A) and (B), x denotes the radian measure of the angle. In (C), the symbol e denotes the base of the Naperian logarithms. The notation $n!$, read "factorial n," denotes the product of all the integers from 1 to n, inclusive.

160. Exponential form of the complex unit.—We now define $\sin(a+bi)$, $\cos(a+bi)$, $e^{(a+bi)}$, by replacing x by $a+bi$ in (A), (B), (C).

Then,

$$\sin(a+bi) = a + bi - \frac{(a+bi)^3}{3!} + \frac{(a+bi)^5}{5!} - \frac{(a+bi)^7}{7!} + \dots$$

$$\cos(a+bi) = 1 - \frac{(a+bi)^2}{2!} + \frac{(a+bi)^4}{4!} - \frac{(a+bi)^6}{6!} + \dots$$

$$e^{(a+bi)} = 1 + \frac{(a+bi)}{1} + \frac{(a+bi)^2}{2!} + \frac{(a+bi)^3}{3!} + \dots$$

If in (C) x is replaced by xi, we have

$$e^{xi} = 1 + xi + \frac{(xi)^2}{2!} + \frac{(xi)^3}{3!} + \frac{(xi)^4}{4!} + \dots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right).$$

The first parenthesis is exactly the right member of (B) and the second parenthesis is the right member of (A).

Hence we have the equality

$$e^{xi} = \cos x + i \sin x.$$

In other words, e^{xi} is a complex unit whose amplitude is x.

161. Exponential form of the sine and cosine.—If in the identity

$$e^{xi} = \cos x + i \sin x \tag{1}$$

x is replaced by $-x$ we have

$$e^{-xi} = \cos x - i \sin x \tag{2}$$

Subtracting (2) from (1) and dividing by $2i$,

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i} \qquad [72]$$

Similarly, adding (1) and (2) and dividing by 2,

$$\cos x = \frac{e^{xi} + e^{-xi}}{2} \qquad [73]$$

These are known as Euler's exponential values of $\sin x$ and $\cos x$.

162. Hyperbolic functions.—Closely allied to Euler's exponential values of $\sin x$ and $\cos x$ are the functions

$$\frac{e^{x} - e^{-x}}{2} \text{ and } \frac{e^{x} + e^{-x}}{2}$$

which are called the **hyperbolic sine** of x and the **hyperbolic cosine** of x, respectively. These functions are denoted by $\sinh x$ and $\cosh x$.

That is,

$$\sinh x = \frac{e^{x} - e^{-x}}{2}, \quad \cosh x = \frac{e^{x} + e^{-x}}{2}.$$

The hyperbolic tangent (tanh), cotangent (coth), secant (sech), and cosecant (csch) are defined by the relations,

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

From

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i} \text{ and } \sinh x = \frac{e^{x} - e^{-x}}{2}$$

it follows that

$$\sin xi = \frac{e^{ixi} - e^{-ixi}}{2i} = \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{i\,(e^{-x} - e^{x})}{2\,(i)^2} = \frac{i\,(e^{x} - e^{-x})}{2},$$

and hence,

$$\sin xi = i \sinh x.$$

EXERCISES

Show that $\cos xi = \cosh x$, $\tan xi = i \tanh x$, $\cot xi = -i \coth x$, $\sec xi = \operatorname{sech} x$, $\csc xi = -i \operatorname{csch} x$.

163. Relations among hyperbolic functions.—From the definitions given in §162, tanh x, coth x, sech x, and csch x may be expressed in forms analogous to those given for sinh x and cosh x. Thus,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\dfrac{e^x - e^{-x}}{2}}{\dfrac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Similarly,

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}, \text{ and}$$

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}.$$

Identities similar to those found for trigonometric functions may now be found for hyperbolic functions.

Example 1. Prove, $\cosh^2 x - \sinh^2 x = 1$.

Proof:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.$$

Example 2. Prove $\sinh 2x = 2 \sinh x \cdot \cosh x$.

Proof:

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2} = 2\,\frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2}$$

$$= 2 \sinh x \cosh x.$$

EXERCISES

Show that the following identities hold among hyperbolic functions:

1. $\sinh(-x) = -\sinh x$.
2. $\cosh(-x) = \cosh x$.
3. $\cosh 2x = \cosh^2 x + \sinh^2 x$.
4. $\operatorname{sech}^2 x + \tanh^2 x = 1$.
5. $\coth^2 x - \operatorname{csch}^2 x = 1$.
6. $e^x = \cosh x + \sinh x$.
7. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.
8. $\coth 2x = \dfrac{\coth^2 x + 1}{2 \coth x}$.
9. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.
10. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.
11. $\sinh x + \sinh y = 2 \sinh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$.
12. $\sinh x - \sinh y = 2 \cosh \dfrac{x + y}{2} \sinh \dfrac{x - y}{2}$.
13. $\cosh x + \cosh y = 2 \cosh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$.
14. $\cosh x - \cosh y = 2 \sinh \dfrac{x + y}{2} \sinh \dfrac{x - y}{2}$.

164. Inverse hyperbolic functions.—Functions similar to the inverse trigonometric functions (see Chapter X) are obtained for hyperbolic functions.

If

$$y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^x - \frac{1}{e^x}}{2} = \frac{e^{2x} - 1}{2e^x},$$

then

$$2\,ye^x = e^{2x} - 1.$$

Solving the quadratic in e^x we have,

$$e^x = y \pm \sqrt{y^2 + 1}.$$

Since e^x is positive, the negative sign is excluded.

Hence,

$$x = \log_e (y + \sqrt{y^2 + 1}).$$

In general mathematical discussions logarithms are always taken to the base e. Hence in practice this base is frequently omitted and the above equation written

$$x = \log (y + \sqrt{y^2 + 1}).$$

If $y = \sinh^{-1} x$, the notation $\sinh^{-1} x$ is used to indicate the inverse function, as is the case in trigonometric functions. That is,

$$\sinh^{-1} y = \log (y + \sqrt{y^2 + 1}).$$

EXERCISES

Prove the following identities:

1. $\cosh^{-1} x = \log (x + \sqrt{x^2 - 1})$.

2. $\tanh^{-1} x = \frac{1}{2} \log \dfrac{1 + x}{1 - x}$.

3. $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x} = \frac{1}{2} \log \dfrac{x + 1}{x - 1}$.

4. $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x} = \log \dfrac{1 + \sqrt{1 - x^2}}{x}$.

5. $\operatorname{csch}^{-1} x = \sinh^{-1} \dfrac{1}{x} = \log \dfrac{1 + \sqrt{1 + x^2}}{x}$.

6. $\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1} \dfrac{x + y}{1 + xy}$.

7. $\sinh^{-1} x = \tanh^{-1} \dfrac{x}{\sqrt{1 + x^2}}$.

8. $\tanh^{-1} \dfrac{2x}{1 + x^2} = 2 \tanh^{-1} x$.

CHAPTER XIII

MISCELLANEOUS SUPPLEMENTARY MATERIAL

In this chapter are given some of the results obtained in more elementary courses in mathematics of which use is made in trigonometry, and also some applications which are of interest to many but which do not naturally form a part of the main body of the text.

165. The quadratic equation and the sign of square roots.— The solution of the quadratic is shown in the following:

Example. Solve $ax^2 + bx + c = 0$.

Multiplying both members of the equation by $4a$, and transposing the last term, gives

$$4a^2x^2 + 4abx = -4ac.$$

Adding b^2 makes the left member a trinomial square,

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$$

Taking square roots and dividing by $2a$,

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}.$$

This formula gives the solution of any quadratic which is reduced to the standard form given above.

By actual substitution it can be shown that each of the results,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

satisfies the original equation. In practical work, however, it is necessary to find by actual test whether one or both of these values gives an answer to the problem which is being solved by means of the quadratic.

Thus on page 66 it was found that

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \quad [24], \text{ and } \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}. \quad [25]$$

Since the sine of an angle in the first and second quadrant is positive, the + sign must be taken in [24] if angle A is between 0° and 360°. If the angle A is between 360° and 720°, or if it is negative and between 0° and − 360°, the negative sign must be taken.

Since the cosine of an angle is positive in the first quadrant and negative in the second, the positive sign must be taken in [25] if A is between 0° and 180°, and the negative sign if A is between 180° and 360°.

In formulas [43], . . . , [46] only the positive square roots are given, since an angle of a triangle is always less than 180°, and hence $\frac{A}{2}$ in these formulas is acute and all its functions are positive. In formulas [49], [50], [51] only the positive roots are given, since the area of a triangle and the radius of a circle are always positive.

166. Proving identities.—On page 17 the identity

$$\frac{1}{\tan A + \cot A} = \sin A \cos A \tag{1}$$

was proved by operating on this equality precisely as is done in solving an ordinary algebraic equation and by substituting from known trigonometric identities. In this way were obtained the equalities

$$1 = \tan A \sin A \cos A + \cot A \sin A \cos A, \tag{2}$$

$$1 = \frac{\sin A}{\cos A} \sin A \cos A + \frac{\cos A}{\sin A} \sin A \cos A, \tag{3}$$

$$1 = \sin^2 A + \cos^2 A, \tag{4}$$

the last of which is a known identity. It was then inferred that (1) is an identity because (4) is.

The validity of this argument depends upon the fact that each operation that has been used is reversible by means of a valid operation. That is, the operation used in passing from (3) to (4) may be reversed, thereby passing from (4) to (3). In this manner we may pass from (3) to (2) and from (2) to (1).

This method of proving identities is always valid except as noted below. Among the operations used in solving algebraic equations there are two which may lead from an untrue to a true statement. One of these is squaring both members of an equation. This is due to the fact that the squares of negative and positive numbers with the same absolute value are equal. If this operation is permitted we may pass from a statement that any two numbers are equal to a true statement.

Thus suppose we write $28 = 4$. Subtracting 16 from both members gives $12 = -12$, and squaring gives $144 = 144$.

The other of these two operations is multiplying both members by an expression which is identically equal to zero.

Thus suppose we write the false statement $\cos x = 2$, and then multiply both members by $\sin^2 x + \cos^2 x - 1$, we obtain $\cos x \sin^2 x + \cos^3 x - \cos x = 2 \sin^2 x + 2 \cos^2 x - 2$, which is a true identity.

It is necessary, therefore, to scrutinize proofs of identities with respect to these two operations. In practice this is a very simple matter and is just what must be done in solving algebraic equations.

It should be remarked also that a true identity may be reduced to an untrue statement by dividing both members by an expression which is identically equal to zero.

167. Theorems from geometry used in trigonometry.—The properties of similar triangles are used in defining the trigonometric functions. (See page 2.) The theorems of which immediate use is made are:

(1) *If the angles of one triangle are equal, respectively, to the angles of another, the triangles are similar.*

(2) *If two triangles are similar, the corresponding sides form a proportion.*

That is, if in two triangles ABC and $A'B'C'$, $\angle A = \angle A'$, $\angle B = \angle B'$ $\angle C = \angle C'$, then the triangles are similar and

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

It is this property of similar triangles which is used in showing that the trigonometric functions of an angle depend upon the angle only, and not upon the particular construction used in defining them.

In finding the values of the functions of 30°, 45°, and 60° and in dealing with complementary angles, use is made of the following propositions either directly or indirectly. (See pages 6, 7, and 8.)

(3) *The sum of the acute angles of a right triangle is equal to one right angle.*

(4) *If in a right triangle one angle is* 30°, *then the side opposite this angle is half the hypotenuse.*

(5) *If two angles of a triangle are equal, the sides opposite them are equal.*

(6) *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the sides.*

(7) *The line from the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base.*

In proving the addition (and subtraction) theorems in trigonometry (see page 64), use is made of the theorem:

(8) *If the two sides of an angle are perpendicular, respectively. to the two sides of another angle, then the angles are equal or supplementary.*

The law of cosines (see page 76) contains, besides the Pythagorean proposition [(6) above], the following two theorems, though in the proof use is made only of the Pythagorean proposition.

(9) *The square of a side opposite an acute angle of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other one upon it.*

(10) *The square of a side opposite an obtuse angle of a triangle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides and the projection of the other one upon it.*

In finding formulas for the area of a triangle and in finding the radii of the inscribed and circumscribed circles the following theorems are used (see pages 82, 83).

(11) *The area of a triangle is equal to one-half of the product of its base and altitude.*

(12) *A tangent to a circle is perpendicular to the radius drawn to the point of tangency.*

(13) *An angle inscribed in a circle is measured by one-half the intercepted arc.*

In solving regular polygons (see page 52) use is made of the theorem:

(14) *A circle may be circumscribed about a regular polygon of any number of sides.*

Use is also made of the fact that the ratio of the circumference of a circle to its diameter, π, is the same for all circles.

168. Closeness of approximation in measuring.—In practical uses of trigonometry the great majority of the numbers that occur in the data of problems are the results of measurements, and these are always subject to certain errors. These errors are due to a variety of causes, among which the following are the most frequent in occurrence:

(1) Measurements are always by means of instruments and no instrument is absolutely correct. This applies equally to the measurement of angles and of lines.

(2) Instruments, and especially steel tapes or chains used in measuring lengths, vary with the temperature and with the tension to which they are subjected.

(3) In applying a chain or tape a number of times in measuring a length, it is difficult to start one application where the preceding one ended.

(4) It is difficult to avoid "zigzagging" when a chain is used a number of times.

(5) Reading angles closely even on a good instrument is very difficult. Thus the sum of the angles of a triangle, where these are obtained by measurement, may easily differ from 180° by 30″ or more.

For these reasons *all* measurements are only approximately correct, though the greatest possible degree of error may be known for any particular measurement.

The degree of possible error varies for different kinds of work.

Thus, in a certain type of surveying it may be known that the error is not greater than five feet in the mile, while in another type of surveying it may be known that the error is not greater than one foot in the mile.

The number of significant figures used to represent the result of a measurement indicates (or at least should indicate) the known closeness of approximation of the measurement. Thus if the length of a line is given as 97.6 feet, this should mean that the length is known to be between 97.55 feet and 97.65 feet; and if a line is given as 240,800 feet, this should indicate that the length is known to lie between 240,750 feet and 240,850 feet. If an angle is given as 25° 37′, this should indicate that the angle is known to lie between 25° 36′ 30″ and 25° 37′ 30″.

169. Approximate character of the tables.—As noted on page 32, nearly all logarithms can be given only approximately by means of decimals (or any other kind of terminating fractions). This is equally true of natural trigonometric functions and of their logarithms. From the table we find, for instance, log 2942 = 3.46864. This means that the logarithm lies between 3.468635 and 3.468645, or may possibly equal the former. That is, this five-place logarithm may be too large or too small by .000005. It is evident that in adding or subtracting logarithms errors may be introduced in the last place. In such problems as occur in trigonometry this error in the last place is seldom greater than 1 and almost never greater than 2.

170. Rules for computing with approximate numbers.—A full treatment of the subject of possible errors in results of computation due to errors in the data would be out of place here. We must therefore content ourselves with the following practical rules.

(1) *The greatest possible error in a sum or a difference is equal to the sum of the greatest possible errors in the numbers added or subtracted.*

(2) *In a product the greatest possible per cent of error is equal to the sum of the greatest possible per cents of errors in the factors.*

(3) *In a quotient the greatest possible per cent of error is the sum of the greatest possible errors in the dividend and divisor.*

(4) *Results obtained from logarithmic computation are usually accurate to as many significant figures as there are places in the table, and are certainly accurate to one less significant figure.*

171. Sources of errors in computation.—The practical surveyor, engineer, or scientific worker must check his results so that he *knows* they are correct to the required degree of accuracy. To check intelligently so as to discover not only incorrect results but also the errors which led to them, it is necessary to know the principal sources of errors.

Gross errors in trigonometric computation are likely to be due to one of the following:

(1) Use of wrong formulas in solving the problem.

(2) Misplacement of the decimal point or, what results in the same error, use of a wrong characteristic.

(3) Wrong operation on logarithms, such as adding instead of subtracting.

(4) Using a wrong column of trigonometric functions, as, for instance, using sines instead of cosines or tangents.

(5) Reading a table of trigonometric functions from the top of the page instead of from the bottom or *vice versa.*

(6) Incorrect addition or subtraction of logarithms near the left end of the numbers.

If the errors are slight so that the checks such as are suggested on pages 72–75 are not quite satisfactory, the error is likely to be one of the following:

(1) Incorrect interpolation.

(2) Adding instead of subtracting *in interpolation* or *vice versa.*

(3) Incorrect addition or subtraction of logarithms in the higher decimals.

172. Checking gross errors by geometric construction.—Suppose it is given that in a right triangle ABC, $\angle A = 42° 23'$, $a = 18.3$ and it is required to find b and c.

Construction: Make construction on a large piece of paper. Construct a right-angle C. Lay off $CB = 18.3$, using a quarter-inch as a unit. With a protractor lay off an angle nearly $42° 23'$ (about $42\frac{1}{2}°$) and complete the triangle. On measuring, it should then be found that AB is about 27 units (27.148) and AC is about 20 units (20.05).

Again, suppose it is given that $\angle A = 68° 30'$, $\angle B = 53° 18'$, $a = 58$. First find $\angle C = 180° - (A + B) = 58° 12'$.

Construction. Using a convenient unit, possibly one-eighth of an inch, lay off $BC = 58$. At the ends of BC lay off angles as nearly as may be equal to $53° 18'$ (53°) and $58° 12'$ (58°), and complete the triangle. On measuring it should then be found that AB and AC are 53 units (52.98) and 50 units (49.98), respectively.

If this check is used, the construction should be made before the work of computing is begun.

173. Finding small angles in exceptional cases.—From the table we find that log cos 0° to log cos 0° 16′ are all given as 0.00000, log cos 0° 17′ to log cos 0° 28′ are all given as 9.99999, log cos 0° 29′ to log cos 0° 36′ are all given as 9.99998, etc. It thus appears that the cosines of very small angles are not well adapted for computation. This applies for the same reason to the sines of angles near 90°. For this reason the cosines of small angles should be avoided, and also the sines of angles near 90°. It may occur, however, that the data given (or obtained from measurement) are such as to lead to the cosine of a small angle.

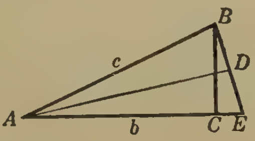
Suppose that in the right triangle ABC, b and c are given and it is required to find the angle A. We then proceed as follows: Construct the bisector of $\angle A$ and draw BE perpendicular to it.

Then $AE = c$, $CE = c - b$, $\angle EAD = \angle CBE$,
and

$$\tan CBE = \frac{CE}{BC} = \frac{c-b}{BC} = \frac{c-b}{\sqrt{c^2-b^2}} = \sqrt{\frac{c-b}{c+b}}.$$

That is,

$$\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}.$$



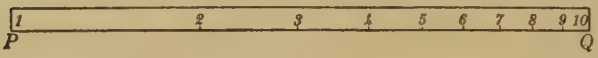
This equation should be used to find $\angle A$.

For angles near 90° a similar formula involving the cotangent should be used.

174. The slide rule.—*The slide rule* is a device by means of which products, quotients, and roots are read off directly to a very considerable degree of accuracy. The principles involved are the same as in logarithms, and the device consists of an adaptation of what is called the *logarithmic scale.*

From a table of logarithms we find that

$\log 1 = 0.00$ $\log 2 = 0.30$ $\log 3 = 0.48$
$\log 4 = 0.60$ $\log 5 = 0.70$ $\log 6 = 0.78$
$\log 7 = 0.85$ $\log 8 = 0.90$ $\log 9 = 0.95$
$\log 10 = 1.00$

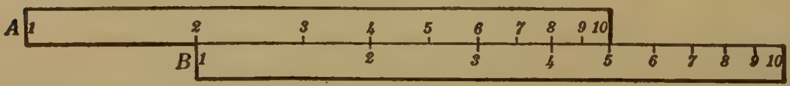


From these numbers a logarithmic scale is constructed by taking a convenient segment PQ as a unit and laying off on it from the left end segments whose lengths are equal to the logarithms of the numbers written above the line.

Thus the left end of the line is marked 1 because $\log 1 = 0$. The distance from P to the point marked 2 is 0.30 because $\log 2 = 0.30$, the distance from P to the point marked 3 is 0.48 because $\log 3 = 0.48$, etc. This scale represents the logarithms of numbers from 0 to 10.

For practical use the scale is extended to twice this length, so as to give the logarithms of numbers from 1 to 100, but the principle used is the same as that of the scale given here.

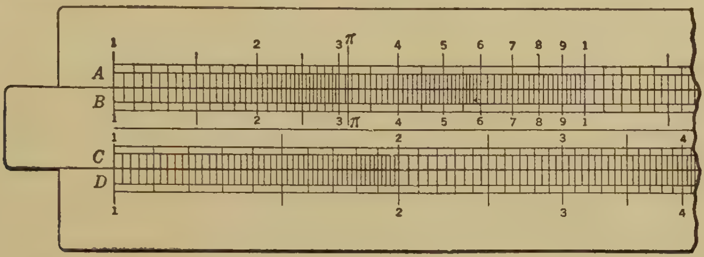
Suppose we have two logarithmic scales, exactly alike, and that they are placed side by side as shown in the figure, so that the numeral 1 in the lower scale B is exactly opposite the numeral 2 in the upper scale A. Then 2 on scale B is opposite 4 on scale A, 3 on scale B is opposite 6 on scale A, etc. This is due to the fact that placing 1 on scale B opposite 2 on scale A, moves scale B a distance to the right equal to log 2. By the theory of logarithms we know that adding log 2 to the logarithm of a number gives the logarithm of the product when that number is multiplied by 2.



If 1 on scale B is placed opposite 3 on scale A, then 2 on B will be opposite 6 on A and 3 on B will be opposite to 9 on A.

To divide 6 by 2 place 2 on scale B opposite 6 on A. Then 1 on B will be opposite 3 on A, and this is the required quotient.

The slide rule consists of a rule carrying two scales usually marked *A* and *D* and a slide carrying two scales marked *B* and *C*. Scales *A* and *B* are exactly alike and are used for finding products and quotients.



The two scales *C* and *D* are also just alike, but the distance from the end point to each number is twice as great as in scales *A* and *B*. It follows that on scale *C* each numeral falls exactly opposite its square on scale *B*. These two scales may therefore be used to find squares and square roots.

Scales *A* and *B* are in direct contact and may be read easily. To compare scales *B* and *C* a device called a *runner*, is useful.

Since scales *C* and *D* are alike, they may be used exactly the same as *A* and *B* to find products and quotients when the numbers involved are small.

The slide may be pulled out entirely and inserted so as to show its reverse side, on which it carries a scale of logarithmic sines marked *S*, and a scale of logarithmic tangents marked *T*. These are used with scales *A* or *D* to find products and quotients when sines and tangents are involved.

Effective use of the slide rule involves many details which cannot be given here. But mastery of the simple principles just stated will make the rest comparatively easy. Detailed rules for using this instrument are usually compiled by the manufacturers and presented to each purchaser.

The results obtained from the ordinary slide rule are sufficiently accurate for many purposes and in all cases they constitute effective checks except for slight errors. Simple inexpensive rules, called students' slide rules, may be obtained. These are useful in learning to use the instrument and for checking.

175. Cumulative reviews.—It is a well-known fact that anything that has been learned, no matter how completely it is mastered at the time of first learning, will soon be forgotten so that complete reproduction is impossible. It may be relearned, however, with less effort than was required for the original learning. To secure comparatively permanent lodgment in the memory a series of relearnings of considerable length is necessary. To meet this requirement a series of cumulative reviews are here provided. In these reviews the principal formulas of trigonometry are made to appear so that each one will be reviewed in accordance with the latest information about the best spacing of such reviews.

Cumulative Review I

(May be used after page 18 of the text)

1. Make the necessary constructions and define the six trigonometric functions of an acute angle.

2. If $\sin A = \frac{2}{3}$ (A being acute), find the other five functions of this angle, (a) by constructing a right triangle and reading the values of the functions directly from it, (b) by using the relations among the functions.

3. Define cofunctions of an angle; state the relations between cofunctions of complementary angles. If the cofunctions of two acute angles are equal, what can be said of the angles?

Find the value of A in each of the following.

4. $\sin 2A = \cos 4A$.
5. $\tan (A - 30°) = \cot (A + 30°)$.
6. $\sec (3A - 15°) = \csc (6A - 30°)$.
7. What trigonometric functions are reciprocals in pairs?

8. Give the values of tangent, cotangent, secant, and cosecant in terms of sine and cosine.

9. Give a relation connecting the tangent and secant of an angle; also one connecting the cotangent and cosecant.

Cumulative Review II

(May be used after page 28 of the text)

1. If $\cos A = \frac{3}{4}$, find the values of the other functions of A.
2. Express the other five functions of an angle in terms of the sine.
3. Express the other five functions of an angle in terms of the secant.
4. What trigonometric functions have their products equal to unity when they are taken in pairs?

5. Show that each function of an angle depends upon the angle and not upon the size of the triangle used in making the definition.

6. Express each of the following as a function of an angle between 0° and 45°: sin 64°, cos 80°, tan 74°, cot 85°, sec 67°, csc 71°.

7. Express each of the following as a function of an angle between 45° and 90°: sin 15°, cos 27°, tan 8°, cot 41°, sec 31°, csc 39°.

In each of the following find the value of x.

8. $\cot (4x - 20°) = \tan (3x + 10°)$.
9. $\csc (2x + 40°) = \sec (3x + 20°)$.
10. $\cos (60° - 2x) = \sin (4x - 10°)$.

Prove the following identities:

11. $\frac{\cos^2 A}{\tan^2 A} = \cos^4 A \csc^2 A$.
12. $\frac{\tan x}{\sec x} = \sin x$.
13. $\sin^2 x \cot x = \sin x \cos x$.
14. $(\sin A - \cos A)^2 = 1 - 2 \sin A \cos A$.
15. $\sin^2 A - \cos^2 A = 1 - 2 \cos^2 A$.
16. $(\tan^2 A + 1) \cos^2 A = 1$.
17. $(\cot^2 A + 1) \sin^2 A = 1$.
18. $\sin^4 A - \cos^4 A = 2 \sin^2 A - 1$.

Cumulative Review III

(May be used after page 46 of the text)

Prove the following identities:

1. $\sin A = \dfrac{1}{\csc A}$.

2. $\cos A = \dfrac{1}{\sec A}$.

3. $\tan A = \dfrac{1}{\cot A}$.

4. $\tan A = \dfrac{\sin A}{\cos A}$.

5. $\cot A = \dfrac{\cos A}{\sin A}$.

6. $\sin^2 A + \cos^2 A = 1$.

7. $\sec^2 A = 1 + \tan^2 A$.

8. $\csc^2 A = 1 + \cot^2 A$.

9. $\tan A \sin A + \cos A = \sec A$.

10. $\dfrac{\tan A \sin A}{\tan A - \sin A} = \dfrac{\sin A}{1 - \cos A}$.

11. $\dfrac{\cos A + \cot A}{\sec A + \tan A} = \cos A \cot A$.

12. $\dfrac{\csc A + 1}{\cot A} = \dfrac{\cot A}{\csc A - 1}$.

13. $\tan^2 A \csc^2 A = 1 + \tan^2 A$.

14. $\cot^4 A + \cot^2 A = \csc^4 A - \csc^2 A$.

15. $\dfrac{\cot A - \cos A}{\cos^2 A} = \dfrac{\cos A \csc A}{1 + \sin A}$.

16. $\tan A + (1 - 2\sin^2 A) \sec A \csc A = \cot A$.

17. $1 + \dfrac{1}{\cot^2 A} = \sec^2 A$.

18. $1 + \dfrac{1}{\tan^2 A} = \csc^2 A$.

19. Express each of the following as a function of an angle between 0° and 45°: sec 78°, csc 49°, sin 56°, cos 83°, tan 64°, cot 69°.

20. In the right triangle ABC, $\angle C = 90°$, and the sides are a, b, c. Give formulas for finding the remaining parts when the following parts are given: a, b; a, c; b, c; A, a; A, b; A, c.

21. Make a list of relations between the trigonometric functions. Give all functions in terms of sine and cosine.

Cumulative Review IV

(May be used after page 56 of the text)

1. If $\sin x = \dfrac{4}{9}$, find the values of the remaining functions of x.

2. Express the other five functions of x in terms of $\cos x$.

3. Find the six functions of each of the angles 30°, 45°, 60°.

4. Define logarithm, characteristic, mantissa, base.

5. Give rules for finding the logarithm of a product, of a quotient, of a power, of a root. State the laws of exponents upon which these rules are based.

6. Multiply and also divide $\bar{2}.84678$ by 3. Find the sum of 1.81906, $\bar{3}.65270$, $\bar{1}.91645$.

Prove the following identities:

7. $\dfrac{\tan x - \sin x}{\sin^2 x} = \dfrac{\sin x \sec x}{1 + \cos x}$.

8. $\dfrac{1}{\csc x - \cot x} = \csc x + \cot x.$

9. $\sec^4 x\,(1 - \sin^4 x) = \sec^2 x + \tan^2 x.$

10. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x} = 2 \csc^2 x.$

11. $\csc^4 x - 1 = 2 \cot^2 x + \cot^4 x.$

12. $\dfrac{\sin x \sec x + \sin y \sec y}{\sin x \sec x - \sin y \sec y} = -\dfrac{\cos x \csc x + \cos y \csc y}{\cos x \csc x - \cos y \csc y}.$

13. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x = 1 + 2 \tan^2 x.$

14. $(\csc x - \cot x)^2 = \dfrac{1 - \cos x}{1 + \cos x}.$

15. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x.$

16. $\cot^2 x - \cos^2 x = \cos^2 x \cot^2 x.$

Cumulative Review V

(May be used after page 62 of the text)

Find the values of the following:

1. $\dfrac{79.4 \times 1.27 \times 0.079}{5.37 \times 0.787}.$

2. $\dfrac{(0.592)^2 \times (0.0276)^3}{8.15 \times (0.917)^4}.$

3. $\dfrac{\sqrt[3]{1.917} \times \sqrt[4]{0.986} \times \sqrt[2]{2.46}}{3.59 \times (0.426)^3}.$

4. $\dfrac{3.4 \times \tan(62° 41') \cos(39° 25')}{9 \sin(42° 16' 20'')}.$

In each of the following find A.

5. $\log \sin A = 9.81426.$

6. $\log \tan A = 0.81914.$

7. $\cos A = \dfrac{19 \sin 41° 47' \times \cos 13° 51'}{24 \cos 51° 32'}.$

8. $\tan A = \dfrac{164 \sin 61° 27' \cot 19° 12'}{37 \tan 39° 37' 40''}.$

9. Define the projection of a segment upon a line; give formulas for finding the projections of a segment AB upon each of the mutually perpendicular axes OX and OY.

10. Express each of the following as a function of an angle between 0° and 45°: sin 49°, cos 56°, tan 62°, cot 57°, sec 74°, csc 46°.

Prove the following identities:

11. $\sin^4 \theta - \cos^4 \theta = (\sin \theta + \cos \theta)(\sin \theta - \cos \theta).$

12. $\dfrac{1}{\cos \theta \csc \theta + \sin \theta \sec \theta} = \sin \theta \cos \theta.$

13. $\dfrac{1 + \sin^2 \theta \sec^2 \theta}{1 + \cos^2 \theta \csc^2 \theta} = \sin^2 \theta \sec^2 \theta.$

14. $\dfrac{\tan \alpha + \tan \beta}{\cot \alpha + \cot \beta} = \tan \alpha \tan \beta.$

15. $\dfrac{\tan \alpha - \tan \beta}{\cot \alpha - \cot \beta} = -\tan \alpha \tan \beta.$

16. $\dfrac{\tan \alpha + \cot \beta}{\sec \alpha - \csc \beta} = \dfrac{\sec \alpha + \csc \beta}{\tan \alpha - \cot \beta}.$

Cumulative Review VI

(May be used after page 69 of the text)

1. Express each of the following as a function of an angle between 0° and 90°: sin 98°, cos 174°, tan 135°, cot 127°, sec 149°, csc 164°.

2. Express each of the following as a function of an angle between 0° and 45°: sin 147°, cos 122°, tan 94°, cot 108°, sec 167°, csc 151°.

3. Find the area of a regular twelve-sided polygon inscribed in a circle whose radius is 12 inches; also find the area of such a polygon circumscribed about such a circle.

4. If x and y are the coördinates of a point P on the terminal side of an angle which is placed as in §82, and if r is the distance from O to P, give the definitions of the six functions in terms of x, y, and r.

Find the values of the following:

5. $\dfrac{31.6 \cos 52^\circ\, 18' \times \sin 41^\circ\, 12'}{\sqrt[3]{9.16 \sin 67^\circ\, 47' \times \tan 62^\circ\, 45'}}$.

6. $\dfrac{(0.462)^3 \times (1.94)^2 \times (0.067)^4}{486}$.

Prove the following identities:

7. $\dfrac{\sin\theta \sec\theta \sin\varphi \sec\varphi + 1}{1 - \sin\theta \sec\theta \sin\varphi \sec\varphi} = \dfrac{\sin\theta \sec\theta + \cos\varphi \csc\varphi}{\cos\varphi \csc\varphi - \sin\theta \sec\theta}$.

8. $\sin\theta \sec\theta + \sin\varphi \sec\varphi = \sin(\theta + \varphi) \sec\theta \sec\varphi$.

9. $\dfrac{\sin x \sec x + \sin y \sec y}{\sin x \sec x - \sin y \sec y} = \sin(x + y) \csc(x - y)$.

10. $\dfrac{\sin x \sec x + \cos y \csc y}{\cos y \csc y - \sin x \sec x} = \cos(x - y) \sec(x + y)$.

11. $\sec^2 A + \csc^2 A = \tan^2 A + \cot^2 A + 2$.

12. $\cos^3 A \tan A + \sin^2 A \cos A \tan A = \sin A$.

Cumulative Review VII

(May be used after page 79 of the text)

1. If $\sin A = \frac{1}{2}$ and $\sin B = \frac{1}{3}$, find $\sin(A + B)$ and also $\cos(A + B)$.

2. If $\tan A = \frac{2}{3}$, find $\cot 2A$.

3. Express the other five functions of A in terms of $\tan A$

4. Give the relations of functions of complementary angles.

5. Prove the relations of functions of supplementary angles.

6. If A, B, C are the angles of a triangle, show that $\sin(A + B) = \sin C$, and also that $\sin \frac{1}{2}(A + B) = \cos \frac{1}{2}C$.

Prove the following identities:

7. $\sin A + \sin 2A = 2 \sin \dfrac{3A}{2} \cos \dfrac{A}{2}$.

8. $\sin 4A - \sin 2A = 2 \sin A \cos 3A$.

9. $\cos 2A + \cos 6A = 2 \cos 4A \cos 2A$.

10. $\cos 7A - \cos A = -2 \sin 4A \sin 3A$.

11. $\dfrac{\sin 75^\circ + \sin 15^\circ}{\cos 75^\circ - \cos 15^\circ} = -\cot 30^\circ.$

12. $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \tan 30^\circ.$

13. $\sin 5A \cos 3A = \frac{1}{2}(\sin 8A + \sin 2A).$

14. $\sin 3A \cos 5A = \frac{1}{2}(\sin 8A - \sin 2A).$

15. $\cos 5A \cos 3A = \frac{1}{2}(\cos 2A + \cos 8A).$

16. $\sin 5A \sin 3A = \frac{1}{2}(\cos 2A - \cos 8A).$

17. $\sin 60^\circ \cos 20^\circ = \frac{1}{2}(\sin 80^\circ + \sin 40^\circ).$

18. $\sin 20^\circ \cos 60^\circ = \frac{1}{2}(\sin 80^\circ - \sin 40^\circ).$

19. $\cos 60^\circ \cos 20^\circ = \frac{1}{2}(\cos 40^\circ + \cos 80^\circ).$

20. $\sin 60^\circ \sin 20^\circ = \frac{1}{2}(\cos 40^\circ - \cos 80^\circ).$

21. $\dfrac{\sin 13A - \sin 9A + \sin 5A - \sin A}{\cos 13A - \cos 9A - \cos 5A + \cos A} = -\cot 4A.$

22. $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0.$

23. $2\sin 2A \cos A + 2\cos 4A \sin A = \sin 5A + \sin A.$

24. $2\cos 2A \cos A - 2\sin 4A \sin A = \cos A + \cos 5A.$

Cumulative Review VIII

(May be used after page 90 of the text)

1. Express each of the following as a function of an angle between 0° and 45°.

$\sin 172^\circ 40'$, $\cos 93^\circ 35'$, $\cot 139^\circ 10'$, $\tan 163^\circ 50'$, $\sec 171^\circ 40'$, $\csc 129^\circ 20'$.

2. If $\tan x = \frac{3}{4}$, find $\tan 2x$.

3. Find $\cot(A + 45^\circ)$ and $\cot(A - 45^\circ)$.

Prove the following identities:

4. $\cot x - \tan x = 2\cot 2x.$

5. $\sin(A - B)\cos B + \cos(A - B)\sin B = \sin A.$

6. $\cos(n - 1)A \cos A - \sin(n - 1)A \sin A = \cos nA.$

7. $\sin A \sec A - \sin B \sec B = \sin(A - B)\sec A \sec B.$

8. $\cos(A + B)\cos A + \sin(A + B)\sin A = \cos B.$

9. $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2 \dfrac{A - B}{2}.$

10. $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2 \dfrac{A - B}{2}.$

11. $\dfrac{\sin 2A}{1 + \cos 2A} = \sin A \sec A.$

12. $\csc A\,(1 - \cos 2A) = 2\sin A$ ~~$\sec A$~~.

13. $\dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \sin \dfrac{A}{2} \sec \dfrac{A}{2}.$

14. $\dfrac{\tan A - \tan B}{\cot A + \tan B} = \tan(A - B)\tan A.$

15. $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$

APPLICATION OF LOGARITHMS TO PROBLEMS IN BUSINESS

Certain problems related to investments are almost impossible of solution without logarithms unless other special tables are used.

176. Simple interest.—The amount S of a principal P at simple interest at r per cent per year for n years is given by the formula

$$S = P + Pnr = P(1 + nr).$$

This formula may be used to obtain the value of P when S, n, and r are given; also to find n or r when the remaining quantities are given.

Example. Find what principal will produce $250 in 2 years at 6% simple interest.

Solution:

$$P = \frac{250}{1 + 2(.06)} = \frac{250}{1.12}$$

$$\begin{aligned} \log 250 &= 2.39794 \\ \log 1.12 &= \underline{0.04922} \\ \log P &= 2.34872 \\ P &= \$223.22 \end{aligned}$$

This problem could of course be solved without logarithms by a simple division.

177. Compound interest.—The amount S of P dollars invested for n years at i per cent per annum compounded yearly is easily seen to be

$$S = P(1 + i)^n$$

From this equality,

$$P = S(1 + i)^{-n}.$$

Example 1. Find the amount of $367.50 at 6% interest compounded annually for 12 years.

Solution:

$$\begin{aligned} S &= 367.50\,(1.06)^{12} \\ \log 367.50 &= 2.56526 \\ 12 \log 1.06 &= \underline{0.30372} \\ \log S &= 2.86898 \\ S &= \$739.57 \end{aligned}$$

Example 2. Find the principal which will produce $1500 in 15 years, interest being 4% compounded annually.

Solution:

$$\begin{aligned} P &= 1500\,(1.04)^{\overset{-15}{\cancel{15}}} \\ \log 1500 &= 3.17609 \\ 15 \log 1.04 &= \underline{0.25545} \\ \log P &= 2.92064 \\ P &= \$832.98 \end{aligned}$$

Example 3. How long must $585 be invested at 7% compounded annually to make the amount $1,000?

Solution:

$$\begin{aligned} 585\,(1.07)^n &= 1000 \\ (1.07)^n &= \frac{1000}{585} \\ n\,(\log 1.07) &= \log 1000 - \log 585 \\ n &= \frac{\log 1000 - \log 585}{\log 1.07} \end{aligned}$$

$$\begin{array}{ll} \log\ 1{,}000 = 3.00000 & \\ \log\quad 585 = \underline{2.76716} & \dfrac{0.23284}{0.02938} = 7.93 \text{ (years)} = 7 \text{ yrs. } 11 \text{ mo. approximately.} \\ \text{numerator} = 0.23284 & \end{array}$$

$\log\ 1.07 = 0.02938$ This last process may be performed by long division.

178. Annuities.—Certain problems in *annuities* lead to formulas which are not adapted as a whole to logarithmic computation, but logarithms are convenient and sometimes necessary in carrying out the computations unless special annuity tables are used.

The amount at the end of n years in a fund into which R is paid at the end of each year at i per cent compounded annually is

$$S = R\,\frac{(1 + i)^n - 1}{i}.$$

The use of logarithms in solving an example of this type is shown in the following:

Example 1. Find the amount which will be accumulated at the end of 10 years by payments of $100 at the end of each year, interest at 5% compounded annually.

Solution:

$$S = 100\,\frac{(1.05)^{10} - 1}{.05}$$

$$\log 1.05^{10} = 10 \log 1.05 = 0.21190$$

$$\therefore\ 1.05^{10} = 1.6289$$

$$S = \frac{1.6289 - 1}{.05} = \frac{.6289}{.05} = 1257.40 \text{ (approximately).}$$

Example 2. How long will it take for the fund in example 1 to amount to \$2500?

Solution:

$$2500 = 100\,\frac{(1.05)^n - 1}{.05}$$

$$25 \times .05 = (1.05)^n - 1$$

$$(1.05)^n = 25 \times .05 + 1 = 2.25$$

$$n \log (1.05) = \log 2.25$$

$$n = \frac{\log 2.25}{\log 1.05} = \frac{0.35218}{0.02119} = 16.6 \text{ (years)}.$$

Inasmuch as the payments are made at the end of each year, we conclude that $n = 17$ (years) with a certain surplus at the end of the last year.

EXERCISES

1. Find the amount of \$750 in 6 years at 4.5% compounded annually.

2. Find the amount of \$500 in 3 years at 8% compounded semi-annually.

3. In how many years will a sum of money double itself at 6% compounded annually?

4. Find the amount at the end of 7 years of an annuity in which payments of \$300 are made at the end of each year, interest being at the rate of 7% compounded annually.

5. Find the amount at the end of 4 years of an annuity in which payments of \$200 are made at the end of each six months, interest at 6% compounded semi-annually.

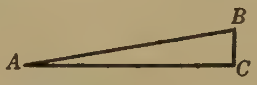
Surveying

Some of the simplest and most common applications of trigonometry are found in ordinary surveying.

While the earth's surface is almost a perfect sphere, small parts of it are so nearly flat that no appreciable errors are introduced by assuming that the figures which are measured are plane.

179. Finding length of course in surveying.—The distances that are used in giving the dimensions of a piece of land are horizontal distances. Thus if a city lot is said to be fifty feet wide, this means that it is fifty feet wide after it has been graded level.

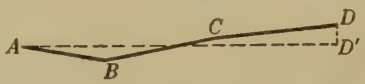
The horizontal distance between two points is called the **course** terminated by these points. Thus if the points A and B are on a hillside, and if ABC is a right triangle with the side AC horizontal and the side BC vertical, then the length of AC is the "course" AB. It is the horizontal distances between points that are shown on maps.



In practice a line of the type AC cannot be measured directly, and is found by measuring AB and the angle CAB and then computing AC by the formula $AC = AB \cos A$.

The angle which the measured line AB makes with the horizontal is called the **incline** of that line. The vertical distance CB of B above or below the point A is called the **rise** or **fall** of AB. CB is computed from the formula $CB = AB \sin A$.

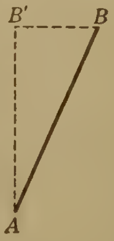
In practice it may be necessary to find the length of course and also the rise or fall between two points A and D when the ground is such that a broken line as shown in the figure must be measured. The surveyor's field notes may contain the entries shown at the left.



Line	Measured Length	Incline
AB	261	$-8°$
BC	384	$+11° 30'$
CD	408	$+5° 45'$

The inclines are marked + and − according as the measured line rises or falls. From these data, the course (the line AD') and the rise or fall ($D'D$) are easily computed.

180. Bearing, latitude, and departure of a course.—Suppose that in the figure, the line AB' runs directly north from A to B', and suppose the angle A is 20°, the line AB extending in a direction 20° east of north. The angle A is indicated by N 20° E, and is said to be the **bearing** of the line AB. This means that if one is standing at A and looking north and then turns through an angle of 20° to the east, he will be looking along the line AB. The expressions N 30° W, S 25° W, S 40° E now have obvious meanings.



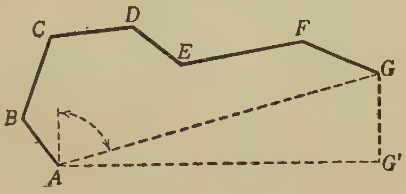
In going from A to B one passes a certain distance northward (or southward). This distance is called the **latitude** of the course. If the end of the course is north of the beginning, the latitude is positive, and if south, the latitude is negative.

In going from A to B one passes a certain distance eastward (or westward). This distance is the **departure** of the course. If the end of the course is east of the beginning, the departure is positive, and if west the departure is negative.

When the bearing and length of a course are given, the latitude and departure are easily computed, the formulas being

latitude = **length of course × cosine of bearing.**
departure = **length of course × sine of bearing.**

181. Finding the closing course of a survey.—It may be necessary to find the latitude and departure of a course such as *AG* in the figure by measuring the courses, *AB*, *BC*, *CD*, *DE*, *EF*, *FG*. The length and bearing of each of these courses are measured and entered in the field notes as shown below.

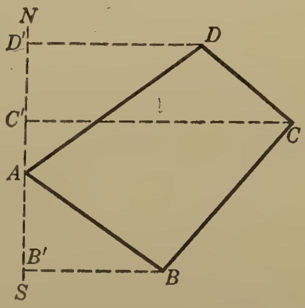


Course	Bearing	Length
AB	N 17° W	693
BC	N 9° E	714
CD	N 83° E	439
DE	S 78° E	314
EF	N 81° E	587
FG	S 79° E	398

The latitude and departure of each course is computed and the net latitude and net departure of *AG* are found. The bearing and length of *AG* are then easily computed. The course *AG* is called the **closing course** of the survey from *A* to *G*.

182. Finding the area of a closed survey.—The area of a polygonal figure may be computed when the length and bearing of each side are known. The method used in practice is called the **double-meridian-distance** (D.M.D.) method. We will now give a brief description of this method.

Suppose it is required to find the area of a quadrilateral *ABCD* shown in the figure. Let *SN* be a line through one vertex of the figure and meeting it at no other point. For convenience *SN* is taken as the north and south line (the meridian) through the most western vertex.[1] The lines BB', CC', and DD' are drawn perpendicular to *SN*. Then we have two triangles $AB'B$ and $AD'D$, and two trapezoids $B'BCC'$ and $C'CDD'$. Clearly the required area is equal to $B'BCC' + C'CDD' - AB'B - AD'D$. The method consists in finding the area of each of these and then the area of the figure.



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[1] In practice this line is determined by the compass and therefore does not run exactly north and south. But the direction of the compass line is essentially the same at all vertices of the figure.

The D.M.D. of a course is the sum of the distances of its end points from the meridian line SN. Thus the D. M. D. of AB is $B'B$, of BC is $B'B + C'C$, etc. It is easily seen that the D. M. D. of a course multiplied by its latitude gives twice the area of the corresponding triangle or trapezoid. The D. M. D.'s of the courses are found in order around the figure, the D. M. D. of any course being found by means of the rule:

The D. M. D. of a course is equal to the D. M. D. of the preceding course, plus the departure of the preceding course, plus the departure of the course itself.

The proof of this rule is left to the student.

This rule must be used, having regard to the sign of the departure of each course. That is, the addition is algebraic. When the figure is taken as suggested above, the D. M. D. of every course is positive.

Example 1. Find the area of the figure $ABCD$, using the data given in the first three columns below.

Course	Bearing	Length	Lat.	Dept.	D.M.D.	Double Areas
AB	S 41° 30′ E	683.2	− 511.69	452.70	452.7	− 231,650
BC	N 57° 20′ E	847.8	457.60	713.69	1619.1	+ 740,900
CD	N 21° 50′ W	1084.0	1006.2	− 403.15	1929.6	+ 1,941,600
DA	S 38° 45′ W	1220.5	− 951.84	− 763.94	762.51	− 725,790

The double areas are found by multiplying the D. M. D.'s by the latitudes. Each result is carried to only five significant figures, for reasons given on page 143. The net area is found by taking the algebraic sum of the double areas and then dividing by 2. If the dimensions are given in feet, find the number of acres by dividing by 43,560, the number of square feet in an acre.

Note that in this example the sums of the positive latitudes and longitudes are almost exactly equal to the sums of the negative latitudes and longitudes. This would be exactly the case if the measurements were exact. In practice these are usually not equal, however, due to inaccuracy of measuring, and certain corrections, called balancing the survey, are necessary.

This will not be considered here, but information about it may be obtained from any of the standard works on surveying.

EXERCISES

In each of the following the courses have the same bearing. Find the length of the course connecting the first and the last point in each example, and also the net rise or fall:

1

Course	Incline	Measured Length
AB	+ 3° 40′	398.4
BC	− 12° 10′	1222.0
CD	+ 6° 20′	1784.0

2

Course	Incline	Measured Length
AB	+ 8° 20′	816.3
BC	+ 13° 10′	513.7
CD	− 7° 30′	1481.0

3

Course	Incline	Measured Length
AB	− 4° 45′	186.7
BC	− 7° 40′	482.3
CD	− 11° 50′	712.5

4

Course	Incline	Measured Length
AB	+ 7° 20′	1983.0
BC	+ 9° 40′	784.2
CD	+ 3° 50′	1167.0

In each of the following find the bearing and the length of the closing course:

5

Course	Bearing	Length of Course
AB	N 49° 30′ W	493.7
BC	N 67° 20′ W	2067.0
CD	S 12° 20′ W	964.2
DA	?	?

6

Course	Bearing	Length of Course
AB	S 17° E	837.6
BC	S 81° 30′ E	928.2
CD	N 16° W	727.9
DA	?	?

7

Course	Bearing	Length of Course
AB	S 54° 40′ W	128.6
BC	S 12° 3′ W	84.7
CD	S 71° E	917.2
DE	N 83° E	346.3
EA	?	?

8

Course	Bearing	Length of Course
AB	N 37° 10′ E	2862.0
BC	S 42° 20′ E	1363.0
CD	S 8° 50′ E	5916
DC	S 21° 40′ W	3815.0
EA	?	?

Find the area in acres in each of the following, the lengths of the courses being given in feet. In each case it is necessary to find first the length and No bearing of the closing course, using the method of §181.

9

Course	Bearing	Length of Course
AB	S 18° 20′ E	817.2
BC	S 83° 10′ E	1262.0
CD	N 6° 50′ W	2281.0
DA		

10

Course	Bearing	Length of Course
AB	S 49° E	176.4
B~~*D*~~C	N 64° 20′ E	854.6
CD	N 71° 10′ W	631.7
DA		

11

Course	Bearing	Length of Course
AB	S 19° 20′ E	217.8
BC	S 68° 30′ E	884.3
CD	N 23° 50′ E	527.6
DE	N 53° 10′ W	391.7
EA		

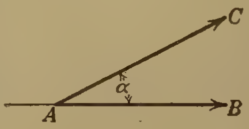
12

Course	Bearing	Length of Course
AB	S 61° E	1984.0
BC	S 48° 50′ E	729.6
CD	N 8° 10′ W	2162.0
DE	N 29° W	317.2
EA		

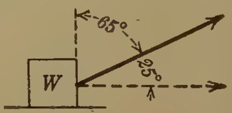
In practice the length and bearing of the closing course are measured directly when that is convenient, since this affords an excellent check on the whole survey. There may, however, be difficulties in measuring this course, such as are suggested in Ex. 3, p. 87. In such cases the method suggested here must be used.

Problems in Physics

183. Components of force.—If a force F is acting along a line AB and if AC makes an angle α with AB, then there is a component of F acting along AC which is equal to $F \cos \alpha$.

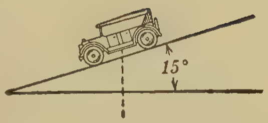


Thus if a weight resting on a horizontal table is subject to a pull of 50 lbs. in a direction making an angle of 25° with the surface of the table, then there is a force of 50 cos 25° tending to drag the weight along the surface of the table, while there is a force of 50 cos 65° tending to lift the weight vertically.

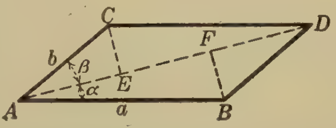


EXERCISES

✓**1.** If an automobile weighing 3500 lbs. is standing on a road which slopes 15°, what force tends to pull it down the hill?

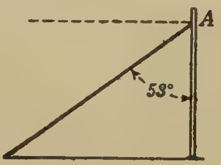


2. Two forces *a* and *b* act in the directions *AB* and *AC* and are represented in magnitude by these segments. Complete the parallelogram *ABDC* and prove that the sum of the components of *a* and *b* along the diagonal *AD* is equal to *AD*.



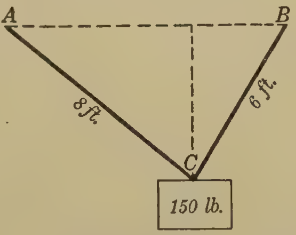
3. A weight of 340 pounds rests on an inclined plane which makes an angle of 7° with the horizontal. What is the force acting at right angles to this plane?

✓**4.** A guy-rope with a breaking strength of 6000 pounds makes an angle of 53° with a post at the point *A*. What part of this strength can be used to withstand a force acting at right angles to the post at this point and in the direction opposite to the side on which the rope is fastened?

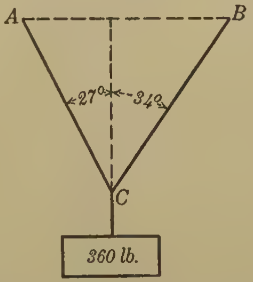


✓**5.** In the figure the points *A* and *B* are in the same horizontal line. If the ropes *AC* and *BC*, supporting a weight of 150 lbs., are 8 feet and 6 feet long, respectively, and if *AB* = 9 feet, what is the strain on each rope?

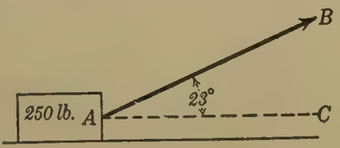
Suggestion. The horizontal component of the two forces must be equal

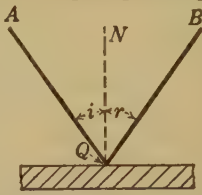


6. Using the dimensions and the weight indicated in the figure, what is the strain on each of the ropes *AC* and *BC*? If the distance *AB* is 7 feet, how much force must be added to the line pulling along *BC* to move the weight one foot to the right? In that case what would be the pull along *AC*?



7. A weight of 250 lbs. rests on a horizontal table, and it is found that a pull of 75 lbs. in the direction *AB* is sufficient to overcome the friction between the weight and the table and start the weight moving. How many pounds of this force are pulling along the line *AC* which is parallel to the table? How many pounds act vertically upward so as to diminish weight resting on the table?





184. Reflection of light.—The path of a ray of light passing through a homogeneous medium is a straight line. When such a ray strikes a polished surface, such as a mirror, it is *reflected* according to the law that

the angle of incidence is equal to the angle of reflection.

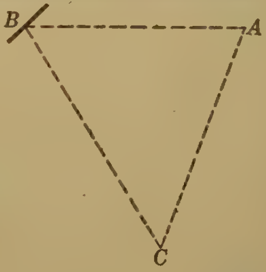
In the figure the ray AQ strikes the reflecting surface at Q and is reflected in the direction QB. If NQ is perpendicular to the surface (NQ is called the *normal* to the surface), then the law of reflection may be stated by saying that AQ and BQ make equal angles with NQ. The angles AQN and BQN are called, respectively, the angle of incidence (i) and the angle of reflection (r).

EXERCISES

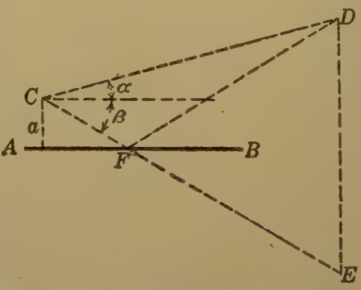
1. Prove that if a mirror is rotated through an angle α, then a reflected ray is rotated through an angle 2α.

2. A light is placed at a point in the perpendicular to the center of a circular mirror and at a distance of 6 feet from it. If the mirror is 10 inches in diameter, find the angle between an incident and reflected ray at the edge of the mirror.

3. In an experiment a mirror is placed at B so that an object at A is seen at C. If $AB = 8.92$, $BC = 9.34$, and $AC = 10.63$, find the angle which the surface of the mirror makes with the line AB.



4. In the figure, AB represents a horizontal reflecting surface (the surface of a smooth lake), C represents the eye of an observer at a vertical distance a above the lake, D represents the top of a mountain, and E the reflection of the mountain in the lake as seen from C. The light from D striking the lake at F is reflected to C and C, F, E are in a straight line. At C the angle of elevation of D is α and the angle of depression of E is β. Find in terms of a, α, and β the distances from C to the mountain.



5. The mirror AB in a dresser swings about a support at C. This mirror is tipped back through an angle α so that the line AD is perpendicular to it. The lines CC' and CA and the angle α are measured and it is required to develop a formula in terms of these quantities giving the length of the line $C'D$.

If CC' is 6.84 feet, $CA = 1.8$ feet, and $\alpha = 11° 30'$, find $C'D$.

Suggestion:

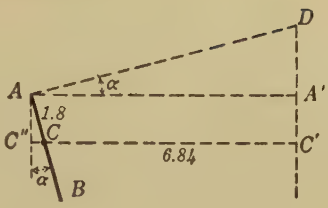
$AA' = CC' + CC'' = CC' + AC \sin \alpha$

and

$C'D = C'A' + A'D = C''A + AA' \tan \alpha$

$= CA \cos \alpha + (CC' + AC \sin \alpha) \tan \alpha.$

6. Find α in the preceding example, if it is given that $CC' = 6.84$, $AC = 1.8$ and $C'D = 4.36$.



Suggestion: Let $C'D = a$, $AC = b$, and $CC' = c$.

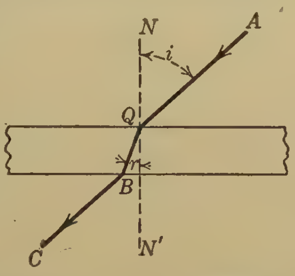
Then $a = b \cos \alpha + (c + b \sin \alpha) \tan \alpha$

or $a = b\sqrt{1 - \sin^2 \alpha} + (c + b \sin \alpha)\dfrac{\sin \alpha}{\sqrt{1 - \sin^2 \alpha}}.$

Then $a\sqrt{1 - \sin^2 \alpha} = b - b \sin^2 \alpha + c \sin \alpha + b \sin^2 \alpha = b + c \sin \alpha.$
Solve for $\sin \alpha$ and substitute values of a, b, c.

185. Refraction of light.—When light passes from one transparent medium into another, it does not generally continue in a straight line, but changes its direction.

Thus if a ray of light is passing through air and meets a glass surface at Q, then that part of the light which is not reflected passes through the glass in the direction QB. If the line NQN' is perpendicular to the glass at Q, then $\angle AQN$ (i) is the **angle of incidence** and BQN' (r) is the **angle of refraction.**



It has been shown by a large number of experiments that for any particular kind of glass, the ratio

$$\frac{\sin i}{\sin r}$$

is the same for all different values of i. This means that no matter how the angle i is changed, the angle r will change in such a way that this ratio will remain unchanged.

This ratio is called the **index of refraction** of one substance with respect to another. For certain kinds of glass and air the value of this index is about 1.5, while for water and air it is about 1.33. That is, the index of refraction of a certain kind of glass with respect to air is 1.5, and of water with respect to air is 1.33.

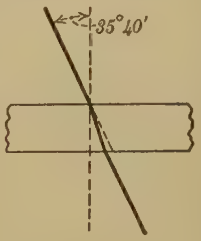
It has been found by very careful experimentation that if a ray passes through a plate of glass whose sides are parallel, then after emerging from the glass the path of the ray will be parallel to its path before entering the glass, that is, in the figure, BC is parallel to AQ.

EXERCISES

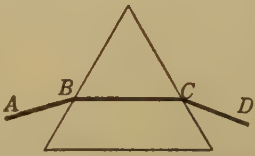
1. In finding the index of refraction of a certain kind of glass with respect to air, i and r were found to be 40° and 24° 42′ respectively. What was the index?

2. Denoting the index of refraction by n, show how to find each one of the quantities i, r, n when the other two are given.

3. A ray of light passes through a glass plate $\frac{3}{8}$ of an inch thick. Find the amount by which this ray is displaced if $i = 35° 40'$ and the index of refraction is 1.49.



4. A ray of light is passed through a triangular prism as shown in the figure. What must be the angle of incidence of the ray AB if the index of refraction is 1.5, if a cross-section of the prism is an equilateral triangle, and if BC is parallel to one side of the prism? Under these same conditions find the angle through which the ray is turned in passing through the prism.

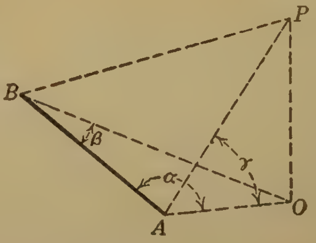


5. A stick partly submerged in water appears to change its direction at the point where it enters the water. What is the apparent direction of the stick under water if its real inclination to the surface is 47° and if the index of refraction of water with respect to air is 1.33?

MISCELLANEOUS THEOREMS AND PROBLEMS

1. The line AB lies in a horizontal plane and P represents a mountain peak visible from both A and B. To find the height of P above the plane of AB and the horizontal distance from A to P, the following measurements are made: $AB = a$, $\angle OAB = \alpha$, $\angle ABO = \beta$, and $\angle OAP = \gamma$. Find formulas giving AO and OP in terms of a, α, β, γ.

Note that while O represents a point in the same horizontal plane with AB and lies inside the mountain, angles α, β, and γ can be measured directly.



2. Two parts of the roof of a house have their ridges at right angles to each other and meet, forming a valley. If each part of the roof has an incline of 45°, what is the incline of the line of intersection of these parts of the roof (the valley)?

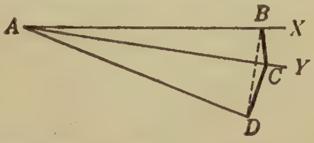
3. A vertical wall is a feet high and stands on a horizontal plane. If α is the altitude of the sun and if the sun is in the same vertical plane with a line at right angles to the wall, what is the width of the shadow?

4. What is the width of the shadow in Example 3 if the sun is in the same vertical plane with a horizontal line which makes an angle β with the wall?

5. A mountain slope makes an angle α with a horizontal plane. Find the incline of a path up the slope if its projection on the horizontal plane makes an angle β with the intersection of the slope and the horizontal plane at its foot.

6. In the preceding example what is the angle between the path and the intersection of the slope with the horizontal plane?

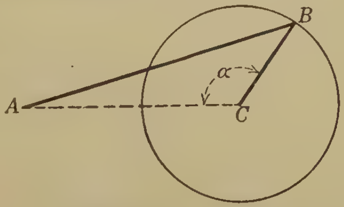
7. In general field-work in geology, and in particular in mining geology, there is a problem of frequent occurrence which amounts to the following: A surveyor is standing at a point A. Along a line AX there is level ground, and somewhat to the right of this line there is a deep canyon in which a point D is visible. The points A, X, D determine a plane and he wishes to find the angle which this plane makes with the horizontal. To do this he measures the angle XAY so that the line AY is vertically above D. Then he measures the angle YAD. If $\angle XAY = \alpha$ and $DAY = \beta$, find a formula giving tan CBD, the plane DCB being perpendicular to the line AX.



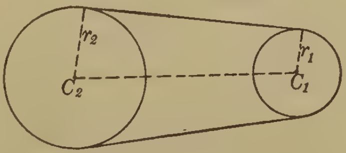
8. A pyramid with a square base of sides a has its apex a distance b above the base. Find the angle between the base and a line running from the apex to a corner of the base.

9. The legs of a tripod are $5\frac{1}{3}$ feet long and open so that the distance between two legs where they meet the ground is $4\frac{1}{2}$ feet. Find the angle which each leg makes with a plumb line from the apex of the tripod. The tripod is assumed to stand on level ground and the legs are assumed to come together at the apex.

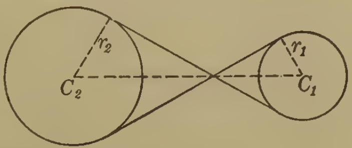
10. In the figure, AB and BC represent the connecting-rod and dish-crank, respectively, of a steam engine. The angle which the dish-crank makes with the line AC is represented by α. Derive a formula giving the length of AC in terms of α, AB, and BC.



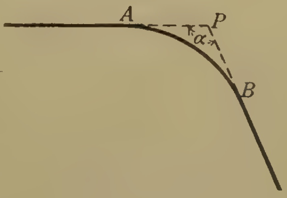
11. In the figure, C_1 and C_2 are the centers of two pulleys with radii r_1 and r_2. Find in terms of r_1, r_2 and C_1C_2 the length of the belt required.



12. Using the same data as in Example 11, find the length of the belt if it is to cross over as shown in the second figure.

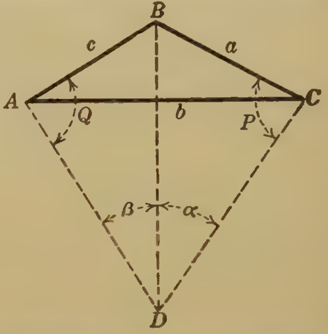


13. Two straight stretches of railway, if extended, would meet at a point P, making an angle α. These two stretches are to be connected by means of a circular arc AB. If the radius of this arc is r, what is the distance from P to A?



14. The length of a degree of longitude on the equator is 60 nautical miles. What is the length of a degree of longitude at latitude α, assuming the earth to be spherical?

15. Assuming the earth to be a sphere with radius r, derive a formula giving the distance from a point the distance h above the surface to the farthest point that can be seen on the surface (the horizon). If the radius of the earth s 3960 miles, how far out at sea can a mountain be seen if it is 15,000 feet high?



16. Three points, A, B, C, are known to be at distances a, b, c from each other as indicated in the figure. From a point D these points subtend angles α and β as indicated. The problem is to find the distances from D to each of the points A, B, and C. (This is called Pothenot's problem.)

Suggestion: Since the distances a, b, c are known, the angles of the triangle ABC may be found and we may regard these as known. Denote the angles DCB and DAB by P and Q, respectively. If these angles can be found, the problem is solved, for then the triangles DCB and DAB can be solved.

By the law of sines we have,

$$DB = a\frac{\sin P}{\sin\alpha} = c\frac{\sin Q}{\sin\beta}.$$

Hence,

$$\frac{\sin P}{\sin Q} = \frac{c\sin\alpha}{a\sin\beta}.$$

and

$$\frac{\sin P - \sin Q}{\sin P + \sin Q} = \frac{c\sin\alpha - a\sin\beta}{c\sin\alpha + a\sin\beta}.$$

By proceeding as in §105 we obtain

$$\frac{\tan\frac{1}{2}(P-Q)}{\tan\frac{1}{2}(P+Q)} = \frac{c\sin\alpha - a\sin\beta}{c\sin\alpha + a\sin\beta}. \tag{1}$$

But

$$P + Q + B + \alpha + \beta = 4 \text{ right } \angle\text{s}$$

and

$$\tfrac{1}{2}(P+Q) = 180^\circ - \tfrac{1}{2}(B+\alpha+\beta).$$

Hence $\frac{1}{2}(P+Q)$ can be found.

Substituting this value in (1), $\frac{1}{2}(P-Q)$ is found.

Since $\frac{1}{2}(P+Q)$ and $\frac{1}{2}(P-Q)$ can be found, P and Q may be found and the problem is solved.

17. If in the preceding example $a = 248.6$, $b = 381.2$, $c = 306.3$, $\alpha = 21^\circ\,35'$, $\beta = 19^\circ\,25'$, find DA, DB, DC.

18. If a projectile is fired at an initial velocity v_0 and at an angle α with the horizontal plane, then, neglecting atmospheric resistance, $y = v_0t\sin\alpha - \frac{1}{2}gt^2$ and $x = v_0t\cos\alpha$, where y is the vertical component of the motion, x the horizontal component, and t the time in seconds from the instant of firing. Find the distance from the gun to the point where the projectile strikes the ground, the plane on which the firing takes place being horizontal.

Suggestion: If $y = 0$, then $t = \dfrac{2v_0}{g}\sin\alpha$.

Substitute this value of t in $x = v_0t\cos\alpha$ and obtain $x = \dfrac{v_0{}^2}{g}\sin 2\alpha$.

What is the value of α which gives the maximum range?

19. Prove that in any triangle

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}, \quad \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}.$$

Suggestion: From $\frac{a}{c} = \frac{\sin A}{\sin C}$ and $\frac{b}{c} = \frac{\sin B}{\sin C}$, we have, by adding,

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\sin C}.$$

But

$$\sin C = \sin (A+B) = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A+B).$$

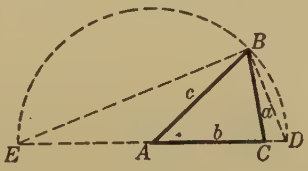
Substituting this value of $\sin C$ and canceling,

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}.$$

The second equality may be found in a similar manner. These formulas are called Mollweide's equations. Since all six elements of a triangle are involved in a single equation, these formulas are often used as checks in computing triangles.

20. A second proof of the law of tangents.

Let ABC be any triangle in which c is greater than b. With A as a center and c as radius, construct a semicircle meeting AC produced in D and E. Join B to D and E.



Then

$$EC = EA + b = c + b,$$
$$CD = AD - b = c - b.$$

Since $AB = AD$, $\angle ABD = \angle ADB$

and

$$\angle ABD + \angle ADB = 180^\circ - \angle A = \angle B + \angle C.$$

Hence,

$$\angle ADB = \angle ABD = \tfrac{1}{2}(B+C),$$

and

$$\angle CBD = \angle ABD - \angle B = \tfrac{1}{2}(B+C) - \angle B = \tfrac{1}{2}(C-B).$$

But $\angle EBD$ is a right angle.

Hence,

$$\angle CEB = 90^\circ - \angle CDB = 90^\circ - \tfrac{1}{2}(C+B), \sin CEB = \cos \tfrac{1}{2}(C+B).$$

Similarly,

$$\sin EBC = \cos \tfrac{1}{2}(C-B).$$

By the law of sines we have in the triangle CBD,

$$\frac{c-b}{a} = \frac{\sin CBD}{\sin CDB} = \frac{\sin \frac{1}{2}(C-B)}{\sin \frac{1}{2}(C+B)}.$$

Similarly from triangle ECB,

$$\frac{c+b}{a} = \frac{\sin EBC}{\sin CEB} = \frac{\cos \frac{1}{2}(C-B)}{\cos \frac{1}{2}(C+B)}.$$

Hence by division,

$$\frac{c-b}{c+b} = \frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)}.$$

CHAPTER XIV

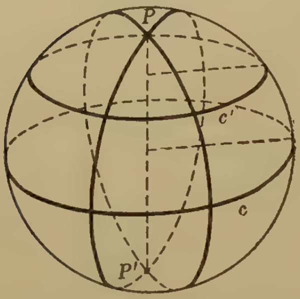
SPHERICAL TRIGONOMETRY

Spherical trigonometry deals with figures which lie on the surface of a sphere. The purpose of this chapter is to study the solution of certain problems related to spherical triangles.

186. Preliminary definitions and theorems.—The following definitions and theorems, which are developed in solid geometry, are given here for convenient reference. They are simple enough to be understood readily by a student who reads them for the first time.

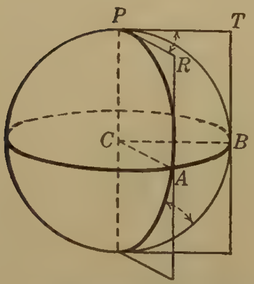
The intersection of a plane with the surface of a sphere is a circle. If the plane passes through the center of the sphere, the circle is a **great circle** of the sphere; otherwise the circle is a **small circle** of the sphere.

A diameter of the sphere perpendicular to the plane of a circle is called the **axis** of the circle. The points in which the axis of a circle meets the sphere are called the **poles** of the circle. In the figure P' and P are poles of each of the circles c' and c.



The angle between two arcs is measured by the angle between tangents to the arcs at a point of intersection.

Thus in the second figure, $\angle APB$ is measured by $\angle RPT$.



A **spherical triangle** is a figure on the surface of a sphere formed by three arcs of three great circles.

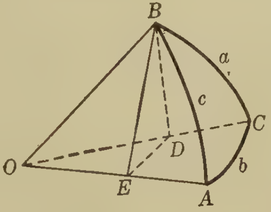
Thus in the figure, APB is a spherical triangle.

Since the sides of such a triangle are arcs, they are measured in degrees or radians, as are also the angles of the triangle.

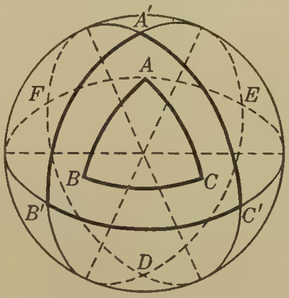
The sum of the sides of a spherical triangle is less than 360°.

The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

187. The spherical triangle and its trihedral angle.—The planes determined by the three sides of a spherical triangle form a trihedral angle whose vertex is at the center of the sphere. If we designate the angles of the spherical triangle by A, B, and C, and the sides opposite them by a, b, and c, as we have done in plane trigonometry, then, O being the center of the sphere, it is evident from the figure that $\angle AOB = c$, $\angle BOC = a$, $\angle COA = b$. Likewise the measure of the dihedral angle A-OB-C is equal to that of angle B; and similarly B-OC-$A = C$ and C-OA-$B = A$.



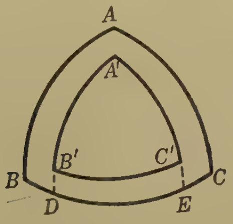
188. Polar triangles.—Using the vertices of a spherical triangle ABC as poles, construct three great circles of the sphere. Denote by A' that point of intersection of the arcs whose poles are B and C which is on the same side of BC as A. Determine B' and C' in a similar way. Then $A'B'C'$ is the **polar triangle** of ABC.

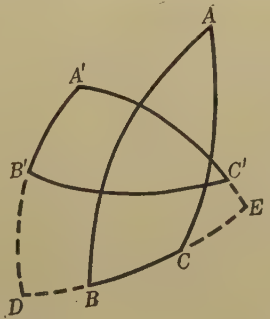


The following theorems are proved in solid geometry.

If one spherical triangle is the polar triangle of a second, then the second is the polar triangle of the first.

An angle of any spherical triangle is the supplement of the opposite side in the polar triangle.





Thus if ABC and $A'B'C'$ are polar triangles and a, b, c and a', b', c' are the corresponding sides, then

$A = 180° - a'$ $A' = 180° - a$

$B = 180° - b'$ $B' = 180° - b$

$C = 180° - c'$ $C' = 180° - c$

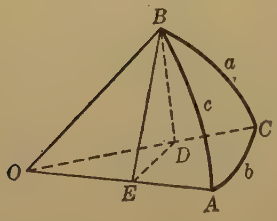
189. Duality in spherical trigonometry.—Suppose that any proposition has been proved for a spherical triangle ABC. Then this same proposition holds of the polar triangle $A'B'C'$ of ABC. Hence from the relations stated in §188, using §187, a proposition is deduced for ABC which corresponds to the original proposition except that "angle" and "side" are interchanged. This principle, known as **duality,** will be used frequently in the following pages.

190. Solving a spherical triangle.—It will appear in this chapter that if three of the six parts of a spherical triangle are given, the triangle is determined. Solving a spherical triangle is the process of finding the three unknown parts when three parts are known.

If the actual lengths of the sides are required, the radius of the sphere must be known. The determination of such length is then a simple process. A problem in spherical trigonometry is regarded as solved when the values of the sides and angles are found in circular measure.

191. Right spherical triangles.—If one of the angles of a spherical triangle is 90°, the triangle is designated as a **right spherical triangle.** In order to solve such a triangle two parts besides the right angle must be given. From the five parts, two known and three unknown, ten combinations of three may be formed. Each of the ten necessary formulas will contain three parts of the triangle. Obviously if any two of these parts are known, the third may be found.

192. Formulas for solving right spherical triangles.—Let ABC be a right spherical triangle whose right angle is at C. Let O be the center of its sphere, the radius of which is unity. Let both a and b be less than 90°. Draw $BD \perp OC$ and $DE \perp OA$. Then BE is $\perp OA$.



Note that in writing, as below, equations like $\angle DOB = a$ and $\angle EOB = c$ we mean to state that the radian measures of the angles and the arcs are the same.

In the figure

$\angle DEB = A,\qquad \angle DOB = a,\qquad \angle EOB = c$

$DB = \sin a,\qquad OD = \cos a,\qquad EB = \sin c,\qquad OE = \cos c$

Then, $ED = DB \cot A = \sin a \cot A$ (1), $ED = EB \cos A = \sin c \cos A$ (2)

$ED = OD \sin b = \cos a \sin b$ (3), $ED = OE \tan b = \cos c \tan b$ (4)

Equating in pairs these values of ED, we obtain the required ten formulas as follows.

From (1) and (2), $\sin a \cot A = \sin c \cos A$,

and hence $\mathbf{\sin a = \sin A \sin c}$ [74]

Then by symmetry, $\mathbf{\sin b = \sin B \sin c}$ [74′]

From (1) and (3), $\sin a \cot A = \cos a \sin b$,

and hence $\mathbf{\sin b = \tan a \cot A}$ [75]

Then by symmetry, $\mathbf{\sin a = \tan b \cot B}$ [75′]

From (1) and (4), $\sin a \cot A = \cos c \tan b$,

and hence $\cos c = \dfrac{\sin a \cot A}{\tan b}$.

Substituting from [75′],

$$\cos c = \frac{\tan b \cot B \cot A}{\tan b},$$

and hence, $\mathbf{\cos c = \cot A \cot B}$ [76]

From (2) and (3), $\sin c \cos A = \cos a \sin b$,

and therefore $\cos A = \dfrac{\cos a \sin b}{\sin c}$.

Substituting from [74′],

$\mathbf{\cos A = \sin B \cos a}$ [77]

and by symmetry $\mathbf{\cos B = \sin A \cos b}$ [77′]

From (2) and (4), $\sin c \cos A = \cos c \tan b$,

or $\mathbf{\cos A = \cot c \tan b}$ [78]

and by symmetry, $\mathbf{\cos B = \cot c \tan a}$ [78′]

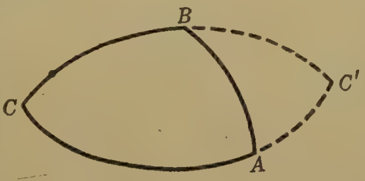
From (3) and (4), $\cos a \sin b = \cos c \tan b$,

and hence $\mathbf{\cos c = \cos a \cos b}$ [79]

EXERCISES

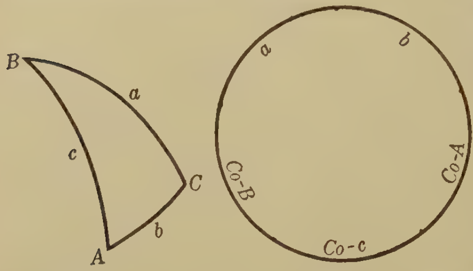
1. Consider a right spherical triangle ABC in which the sides a and b are both greater than 90°, C being a right angle.

Complete the lune $CAC'B$ as shown in the figure. Then in the triangle ABC' the sides about the right angle are less than 90° and the formulas of §192 are known to hold. Using the relations of trigonometric functions for supplementary angles given in §87 prove these ten formulas for the triangle ABC.



2. Construct the figure and prove these propositions for the case when $a<90°$ and $b>90°$.

193. Napier's rules of circular parts.—Two rules devised by Napier help in remembering the ten formulas stated in §192. In a right spherical triangle ABC, with sides a, b, c, let C be the right angle. The so-called circular parts of this triangle are the sides a and b, and the complements of the angles A and B and of the side c. Denote these complements by $Co\text{-}A$, $Co\text{-}B$ and $Co\text{-}c$ respectively and arrange the circular parts in order as indicated in the circle.



Corresponding to any one of the circular parts there are two **adjacent parts** and two others which are referred to as **opposite parts.**

Then Napier's rules are:

(1) *The sine of any circular part is equal to the product of the tangents of the two adjacents circular parts.*

(2) *The sine of any circular part is equal to the product of the cosines of the two opposite circular parts.*

Applying rule (1) to the circular part b, for example, we have

$$\sin b = \tan (Co\text{-}A) \tan a$$

or

$$\sin b = \tan a \cot A.$$

From rule (2),

$$\sin b = \cos (Co\text{-}c) \cos (Co\text{-}B)$$

or

$$\sin b = \sin c \sin B.$$

These are formulas [75] and [74′].

The repetition of *a* in the words *tangent* and *adjacent*, and of *o* in *cosine* and *opposite*, will be a further aid to the memory.

Exercise: Develop each of the formulas of §192 by using Napier's rules.

194. Species.—If each of two parts of a spherical triangle is less than 90°, or each greater than 90°, then the parts are said to be of the **same species.** Formulas [74] to [79] taken as a whole determine whether a part found is in the first or second quadrant, but in an individual case, a part found from its sine is indeterminate. The following rules determine the question of quadrant for all cases in right triangles except one. This case, the ambiguous case, is considered in the next paragraph.

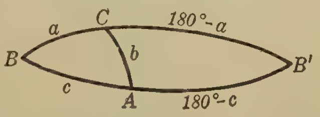
Rule I: *The hypotenuse is less than 90° when the sides about the right angle are of the same species, otherwise the hypotenuse is greater than 90° and conversely.*

For by [79], $\cos c = \cos a \cos b$. Hence, when a and b are in the same quadrant, $\cos c$ is positive; otherwise it is negative. Conversely if $\cos c$ is positive, a and b are of the same species.

Rule II: *An oblique angle and its opposite side are always of the same species.*

For by [77′], $\sin A = \cos B/\cos b$. Hence $\cos B$ and $\cos b$ are of like sign; that is, B and b are of the same species.

195. The ambiguous case.—If the parts given are a side and its opposite angle, the remaining parts are found from their sines. Hence there are two supplementary values for each. The relation of the two triangles is shown in the figure for the case in which B and b are the given parts. B and B' are equal, BB' being a lune.



EXERCISE

Form the equations to be used in solving the spherical triangle, ABC, b, B, and C $(= 90°)$ being given.

196. Use of logarithms.—The formulas developed for solving spherical triangles are all suitable for logarithmic computation and the computation is carried out as in plane trigonometry. The following should be noted:

When negative values occur, the numerical results may still be found by using logarithms, the quality being determined by the number of negative factors. In such cases, the letter n may well be placed after any logarithm, the antilogarithm of which is negative. The quality of the result is then immediately determinable. In the solution of Example 3 on page 175 a suitable form is shown.

When the characteristic of a logarithm is negative, -10 may be written after the mantissa. In this work there is seldom any ambiguity in such cases. It is therefore possible to omit the -10 in all cases. In the examples worked out in the following pages this procedure has been adopted.

197. Solving right triangles.—If two parts of a spherical triangle are given, formulas involving each of the unknown parts with both of the known parts can be formed by the use of Napier's rules. From these the unknown parts may be found, and then as a check on these, the formula involving all three of them may be applied. The quadrant in which each magnitude lies may be determined by the principles of §194 if the formula used leaves any ambiguity. The form of the work will be clear from illustrative examples.

Example 1. Solve the spherical triangle ABC given: $C = 90°$, $c = 78° 25'$, $A = 36° 42' 30''$.

Solution: Using formulas from §192 (or Napier's rules), $\sin a = \sin A \sin c$ from [**74**], $\tan b = \cos A \tan c$ from [**78**], $\cot B = \cos c \tan A$ from [**76**], $\sin a = \tan b \cot B$ from [**75'**] as a check.

$$\begin{array}{rl@{\qquad\qquad}rl}
\log\sin A = & 9.77652 & \log\cos c = & 9.30275 \\
\log\sin c = & \underline{9.99106} & \log\tan A = & \underline{9.87251} \\
\log\sin a = & 9.76758 & \log\cot B = & 9.17526 \\
a = & 35° 50' 37'' & B = & 81° 29' 6'' \\
\text{or} & 144° 9' 23'' & &
\end{array}$$

$$\begin{array}{rl@{\qquad\qquad}rl}
\log\cos A = & 9.90401 & & \textit{check} \\
\log\tan c = & \underline{0.68832} & \log\tan b = & 0.59233 \\
\log\tan b = & 0.59233 & \log\cot B = & \underline{9.17526} \\
b = & 75° 39' 31'' & \log\sin a = & 9.76759
\end{array}$$

Since $c < 90°$ and $b < 90°$, $a < 90°$. Hence the unknown parts in the triangle are $a = 35° 50' 37''$, $B = 81° 29' 6''$, $b = 75° 39' 31''$.

Example 2. Solve the right spherical triangle ABC, given the following: $C = 90°$, $b = 136° 35'$, $B = 107° 21'$.

Solution. The formulas to be used are

$$\begin{array}{rl@{\qquad\qquad}rl}
\sin a = & \tan b \cot B & \sin A = & \cos B/\cos b \\
\sin c = & \sin b/\sin B & (\textit{check})\ \sin a = & \sin A \sin c \\
\log\tan b = & 9.97598 & \log\cos B = & 9.47452 \\
\log\cot B = & \underline{9.49474} & \log\cos b = & \underline{9.86116} \\
\log\sin a = & 9.47072 & \log\sin A = & 9.61336 \\
a = & 17° 11' 40'' & A = & 24° 14' 21'' \\
\text{or} & 162° 48' 20'' & \text{or} & 155° 45' 39''
\end{array}$$

$$\begin{array}{rl@{\qquad\qquad}rl}
\log\sin b = & 9.83715 & & \textit{check} \\
\log\sin B = & \underline{9.97978} & \log\sin A = & 9.61336 \\
\log\sin c = & 9.85737 & \log\sin c = & \underline{9.85737} \\
c = & 46° 3' 35'' & \log\sin a = & 9.47073 \\
\text{or} & 133° 56' 25'' & &
\end{array}$$

When the smaller value of c is used then a and b are of the same species; when the larger value is used, a and b are of different species. In both cases A must be of the same species as a. Hence the results are

$$\begin{cases} c_1 = 46^\circ\ 3'\ 35'' \\ a_1 = 162^\circ\ 48'\ 20'' \\ A_1 = 155^\circ\ 45'\ 39'' \end{cases} \quad \text{and} \quad \begin{cases} c_2 = 133^\circ\ 56'\ 25'' \\ a_2 = 17^\circ\ 11'\ 40'' \\ A_2 = 24^\circ\ 14'\ 21'' \end{cases}$$

Example 3. Solve the right spherical triangle ABC given $c = 112^\circ\ 25'$, $A = 36^\circ\ 42'\ 30''$.

Solution:

$$\begin{array}{rl@{\qquad\qquad}rl}
\sin a = & \sin A \sin c & \tan b = & \cos A \tan c \\
\cot B = & \cos c \tan A & (\textit{check})\ \sin a = & \tan b \cot B \\
\log \sin A = & 9.77652 & \log \cos A = & 9.90401 \\
\log \sin c = & \underline{9.96588} & \log \tan c = & \underline{0.38456}\quad (\text{n}) \\
\log \sin a = & 9.74240 & \log \tan b = & 0.28857\quad (\text{n}) \\
a = & 33^\circ\ 32'\ 41'' & b = & 62^\circ\ 46'\ 19'' \\
\text{or} & 146^\circ\ 27'\ 19'' & &
\end{array}$$

By §194, the former is not admissible.

$$\begin{array}{rl@{\qquad\qquad}rl}
\log \cos c = & 9.58131\quad (\text{n}) & & \textit{check} \\
\log \tan A = & \underline{9.87251} & \log \tan b = & 0.28857 \\
\log \cot B = & 9.45382\quad (\text{n}) & \log \cot B = & \underline{9.45382} \\
B = & 105^\circ\ 52'\ 19'' & \log \sin a = & 9.74239
\end{array}$$

EXERCISES

Solve the following triangles, C in each case being 90°.

Group A

1. $\begin{cases} a = 28^\circ\ 41' \\ B = 71^\circ\ 32' \end{cases}$
2. $\begin{cases} a = 81^\circ\ 44' \\ b = 42^\circ\ 12' \end{cases}$
3. $\begin{cases} A = 21^\circ\ 19' \\ b = 112^\circ\ 15' \end{cases}$
4. $\begin{cases} a = 121^\circ\ 48' \\ c = 50^\circ\ 29' \end{cases}$
5. $\begin{cases} b = 29^\circ\ 55' \\ B = 42^\circ\ 18' \end{cases}$
6. $\begin{cases} c = 82^\circ\ 38' \\ A = 36^\circ\ 11' \end{cases}$
7. $\begin{cases} b = 117^\circ\ 30' \\ c = 83^\circ\ 48' \end{cases}$
8. $\begin{cases} B = 20^\circ\ 58' \\ c = 45^\circ\ 13' \end{cases}$
9. $\begin{cases} b = 111^\circ\ 33' \\ A = 49^\circ\ 9' \end{cases}$
10. $\begin{cases} c = 12^\circ\ 18' \\ B = 67^\circ\ 21' \end{cases}$
11. $\begin{cases} A = 62^\circ\ 35' \\ c = 71^\circ\ 17' \end{cases}$
12. $\begin{cases} a = 39^\circ\ 20' \\ b = 70^\circ\ 29' \end{cases}$

Group B

1. $\begin{cases} b = 98^\circ\ 8' \\ c = 77^\circ\ 42' \end{cases}$
2. $\begin{cases} B = 61^\circ\ 21' \\ a = 76^\circ\ 14' \end{cases}$
3. $\begin{cases} a = 49^\circ\ 45' \\ b = 61^\circ\ 18' \end{cases}$
4. $\begin{cases} b = 37^\circ\ 40' \\ A = 77^\circ\ 24' \end{cases}$
5. $\begin{cases} c = 120^\circ\ 18' \\ B = 67^\circ\ 48' \end{cases}$
6. $\begin{cases} a = 48^\circ\ 47' \\ c = 74^\circ\ 51' \end{cases}$
7. $\begin{cases} c = 79^\circ\ 50' \\ B = 49^\circ\ 7' \end{cases}$
8. $\begin{cases} a = 28^\circ\ 45' \\ A = 64^\circ\ 20' \end{cases}$
9. $\begin{cases} b = 53^\circ\ 15' \\ a = 59^\circ\ 45' \end{cases}$
10. $\begin{cases} B = 70^\circ\ 40' \\ c = 79^\circ\ 13' \end{cases}$
11. $\begin{cases} A = 80^\circ\ 17' \\ c = 85^\circ\ 25' \end{cases}$
12. $\begin{cases} A = 74^\circ\ 5' \\ B = 62^\circ\ 55' \end{cases}$

198. Quadrantal and isosceles triangles.—A **quadrantal spherical** triangle is one which has a side equal to 90°. Hence the polar triangle of a quadrantal triangle is a right triangle. Thus, in order to solve a quadrantal triangle, the polar triangle may be solved, and the values of the unknown parts of the quadrantal triangle are then known immediately.

An **isosceles spherical** triangle has two sides equal The arc drawn from the vertex perpendicular to the base bisects the base and the vertical angle, forming two equal right spherical triangles. If three parts of the isosceles triangle are given, three parts of the right triangles are known, hence the remaining parts may all be found.

EXERCISES

Solve the triangles of which parts are given.

1. $\begin{cases} a = 51^\circ\ 31' \\ b = 63^\circ\ 20' \\ c = 90^\circ \end{cases}$
2. $\begin{cases} a = 38^\circ\ 41' \\ B = 77^\circ\ 12' \\ c = 90^\circ \end{cases}$
3. $\begin{cases} b = 82^\circ\ 25' \\ C = 68^\circ\ 35' \\ c = 90^\circ \end{cases}$
4. $\begin{cases} b = 96^\circ\ 45' \\ B = 38^\circ\ 17' \\ c = 90^\circ \end{cases}$
5. $\begin{cases} B = 18^\circ\ 50' \\ C = 48^\circ\ 29' \\ c = 90^\circ \end{cases}$
6. $\begin{cases} B = 65^\circ\ 11' \\ A = 85^\circ\ 22' \\ c = 90^\circ \end{cases}$
7. $\begin{cases} b = 71^\circ\ 13' \\ A = 118^\circ\ 30' \\ c = 90^\circ \end{cases}$
8. $\begin{cases} a = 112^\circ \\ b = 90^\circ \\ c = 90^\circ \end{cases}$
9. $\begin{cases} b = 53^\circ\ 55' \\ B = 77^\circ\ 50' \\ c = 90^\circ \end{cases}$
10. $\begin{cases} a = 37^\circ\ 35' \\ C = 82^\circ\ 12' \\ c = 90^\circ \end{cases}$
11. $\begin{cases} A = 119^\circ\ 30' \\ B = 45^\circ\ 40' \\ c = 90^\circ \end{cases}$
12. $\begin{cases} a = 100^\circ\ 40' \\ b = 112^\circ\ 19' \\ c = 90^\circ \end{cases}$

ISOSCELES TRIANGLES

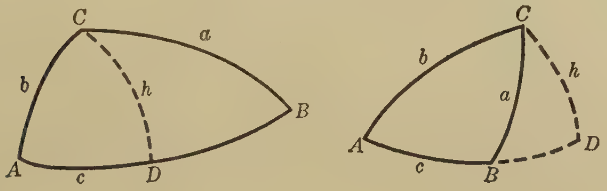
In the following △s, $a = b$ and $A = B$. Find the parts not given.

1. $\begin{cases} b = 24^\circ\ 48' \\ B = 39^\circ\ 22' \end{cases}$
2. $\begin{cases} a = 82^\circ\ 15' \\ C = 130^\circ\ 24' \end{cases}$
3. $\begin{cases} b = 78^\circ\ 18' \\ B = 98^\circ\ 36' \end{cases}$
4. $\begin{cases} a = 16^\circ\ 38' \\ c = 30^\circ\ 46' \end{cases}$
5. $\begin{cases} a = 72^\circ\ 50' \\ A = 56^\circ\ 50' \end{cases}$
6. $\begin{cases} b = 19^\circ\ 42' \\ c = 46^\circ\ 20' \end{cases}$
7. $\begin{cases} b = 59^\circ\ 19' \\ C = 69^\circ\ 29' \end{cases}$
8. $\begin{cases} c = 99^\circ\ 10' \\ C = 78^\circ\ 24' \end{cases}$
9. $\begin{cases} a = 58^\circ\ 27' \\ B = 26^\circ\ 43' \end{cases}$
10. $\begin{cases} c = 48^\circ\ 42' \\ C = 108^\circ\ 18' \end{cases}$
11. $\begin{cases} A = 72^\circ\ 19' \\ C = 55^\circ\ 26' \end{cases}$
12. $\begin{cases} a = 42^\circ\ 38' \\ A = 59^\circ\ 21' \end{cases}$

199. Solution of general spherical triangles.—For spherical triangles in general three parts must be given to determine the triangle. Formulas for finding the remaining parts when any three parts are given will be developed in the following paragraphs.

200. The law of sines.—*The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

In symbols, $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \qquad [80]$$



Proof: In the triangle ABC (either figure) drop the perpendicular h from the vertex C to the opposite side AB (extended, if necessary).

Then by Napier's rules or [74], in the right triangle ADC,

$$\sin h = \sin b \sin A,$$

and in triangle CDB,

$$\sin h = \sin a \sin B.$$

Hence $$\sin b \sin A = \sin a \sin B,$$

or

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

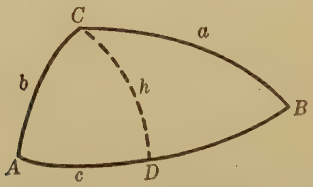
Similarly we may prove

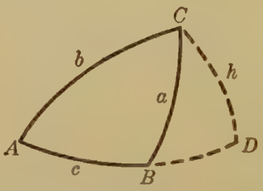
$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C} \text{ and } \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

This law may be used for finding one of the remaining parts of any spherical triangle when two *opposite* parts of the triangle are among the three that are known.

201. The law of cosines.—(a) *The cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of their included angle,* and

(b) *The cosine of any angle of a spherical triangle is equal to minus the product of the cosines of the other two angles plus the product of the sines of these two angles and the cosine of their included side.*





Proof of (a): It is evident from the figure that when direction is considered

$$c = AD + DB, \text{ or } DB = c - AD.$$

From Napier's rules, or [78],

$$\cos A = \tan AD \cot b$$

and hence $$\tan AD = \tan b \cos A.$$

Similarly $$\cos b = \cos h \cos AD$$

and hence $$\cos AD = \frac{\cos b}{\cos h}.$$

But $$\sin AD = \tan AD \cos AD$$
$$= \tan b \cos A \cos b / \cos h.$$

Also
$$\begin{aligned}\cos a &= \cos h \cos DB \\ &= \cos h \cos (c - AD) \\ &= \cos h \cos c \cos AD + \cos h \sin c \sin AD.\end{aligned}$$

Substituting in this last equation for cos AD and sin AD,

$$\cos a = \frac{\cos h \cos c \cos b}{\cos h} + \frac{\cos h \sin c \tan b \cos A \cos b}{\cos h},$$

or $$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad [81]$$

Similarly,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \qquad [81']$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad [81'']$$

This is the law of cosines for sides.

Proof of (b): If $A'B'C'$ is the polar triangle of any given triangle ABC, then formulas [81] hold for $A'B'C'$.

That is, $\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'$.

But $\cos a' = -\cos A$, $\cos b' = -\cos B$, $\cos c' = -\cos C$,

$\sin b' = \sin B$, $\sin C' = \sin C$, and $\cos A' = -\cos a$.

Hence $$\boldsymbol{\cos A = -\cos B \cos C + \sin B \sin C \cos a} \qquad [82]$$

Similarly, $$\boldsymbol{\cos B = -\cos C \cos A + \sin C \sin A \cos b} \qquad [82']$$

$$\boldsymbol{\cos C = -\cos A \cos B + \sin A \sin B \cos c} \qquad [82'']$$

This is the **law of cosines for angles.**

202. The half-angle formulas.—From [81],

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence $$1 - \cos A = \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c} \qquad \text{by [21]}$$

$$= \frac{2 \sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c} \qquad \text{by [35]}$$

If we let $a + b + c = 2s$, so that $\frac{1}{2} (a + b - c) = s - c$, $\frac{1}{2} (a + c - b) = s - b$ and $\frac{1}{2} (b + c - a) = s - a$, we have

$$1 - \cos A = \frac{2 \sin (s - b) \sin (s - c)}{\sin b \sin c}.$$

Similarly $$1 + \cos A = \frac{\sin b \sin c + \cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c}$$

$$= \frac{2 \sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (b + c - a)}{\sin b \sin c}$$

$$= \frac{2 \sin s \sin (s - a)}{\sin b \sin c}.$$

But $\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}$ and $\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}$. (§109)

Hence,

$$\sin^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \tag{1}$$

and

$$\cos^2\frac{A}{2} = \frac{\sin s \sin(s-a)}{\sin b \sin c}. \tag{2}$$

By dividing these,

$$\tan^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}$$

and

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$$

$$= \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

If we now denote this radical by r, we obtain

$$\tan\frac{A}{2} = \frac{r}{\sin(s-a)} \tag*{[83]}$$

Similarly

$$\tan\frac{B}{2} = \frac{r}{\sin(s-b)} \tag*{[83′]}$$

and

$$\tan\frac{C}{2} = \frac{r}{\sin(s-c)} \tag*{[83″]}$$

where

$$r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

From these formulas, the three angles may be found when the three sides are given.

203. The half-side formulas.—If A', B', C', a', b', c' are the parts of the polar triangle of ABC, then by [83]

$$\tan\tfrac{1}{2}\quad A' = \frac{r'}{\sin(s'-a')} \tag{1}$$

where

$$r' = \sqrt{\frac{\sin(s'-a')\sin(s'-b')\sin(s'-c')}{\sin s'}}$$

and

$$s' = \tfrac{1}{2}(a'+b'+c').$$

Then $s' = \frac{1}{2}(180° - A + 180° - B + 180° - C) = 270° - \frac{1}{2}(A + B + C) = 270° - S$,

where $S = \frac{1}{2}(A + B + C)$.

Then $s' - a' = 270° - S - 180° + A = 90° - (S - A)$, and similar relations for b' and c'. Then we have

$$r' = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}},$$

which we will denote by $\frac{1}{R}$.

Furthermore,

$$\tan \tfrac{1}{2} A' = \tan\left(90° - \frac{a}{2}\right) = \cot \frac{a}{2}.$$

Substituting in (1) above,

$$\cot \tfrac{1}{2} a = \frac{1}{R \cos(S-A)}$$

or $$\tan \tfrac{1}{2} a = R \cos(S-A) \qquad [84]$$

Similarly $$\tan \tfrac{1}{2} b = R \cos(S-B) \qquad [84']$$

and $$\tan \tfrac{1}{2} c = R \cos(S-C) \qquad [84'']$$

where $$R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}.$$

In this expression, $\cos S$ is negative since S is always greater than 90° and less than 270°.

Since $a' < b' + c'$,

$$180° - A < 180° - B + 180° - C$$

and $$B + C - A < 180° \text{ so that } S - A < 90°.$$

But $$B + C - A > -180°,$$

so that $$S - A > -90°.$$

Hence $\cos(S-A)$ is positive. In a similar way it is seen that $\cos(S-B)$ and $\cos(S-C)$ are also positive. The value of R is accordingly real in all cases.

Formulas [84] to [84″] are sufficient for finding the three sides when the three angles are given.

204. Napier's analogies.—Dividing $\tan \frac{A}{2}$ by $\tan \frac{B}{2}$ in [83] and [83′],

$$\frac{\tan \frac{A}{2}}{\tan \frac{B}{2}} = \frac{\sin(s-b)}{\sin(s-a)}.$$

Taking this proportion by composition and division,

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)}.$$

Substituting for tangent in terms of sine and cosine, and applying formulas [32] and [33] to the right member,

$$\frac{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A}{2}\sin\frac{B}{2}}=\frac{\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)}.$$

That is,

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{c}{2}}{\tan\frac{1}{2}(a-b)}\qquad[85]$$

Again, multiplying $\tan\frac{A}{2}$ by $\tan\frac{B}{2}$ in [83] and [83′],

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{\sin(s-c)}{\sin s},$$

or

$$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{\sin(s-c)}{\sin s}.$$

Then taking the proportion by division and composition and applying [32] and [33],

$$\frac{\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{\sin s-\sin(s-c)}{\sin s+\sin(s-c)}$$

$$=\frac{\cos\frac{1}{2}(2s-c)\sin\frac{c}{2}}{\sin\frac{1}{2}(2s-c)\cos\frac{c}{2}}.$$

That is,

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{c}{2}}{\tan\frac{1}{2}(a+b)}\qquad[86]$$

By using the values of $\tan \frac{a}{2}$ and $\tan \frac{b}{2}$ from [84] and [84′], and proceeding in a similar manner we obtain

$$\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{C}{2}}{\tan \frac{1}{2}(A-B)} \qquad [85']$$

$$\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\cot \frac{C}{2}}{\tan \frac{1}{2}(A+B)} \qquad [86']$$

The details of the proof of these two formulas is left to the student.

Formulas ~~[84]~~ 85., [86], [85′], and [86′] are known as **Napier's analogies.** From these, the third side and third angle can be found when two sides and the angles opposite are known.

205. Gauss's equations, or Delambre's analogies.—From (1) and (2) of §202 we obtain

$$\sin \frac{A}{2} \cos \frac{B}{2} = \frac{\sin (s-b)}{\sin c} \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}$$

$$= \frac{\sin (s-b)}{\sin c} \cos \frac{C}{2}$$

and

$$\cos \frac{A}{2} \sin \frac{B}{2} = \frac{\sin (s-a)}{\sin c} \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}$$

$$= \frac{\sin (s-a)}{\sin c} \cos \frac{C}{2}.$$

Adding and applying [18] and [33] we have,

$$\sin \frac{A+B}{2} = \frac{\sin (s-a) + \sin (s-b)}{\sin c} \cos \frac{C}{2}$$

$$= \frac{2 \sin \frac{1}{2}(2s-a-b) \cos \frac{1}{2}(b-a)}{2 \sin \frac{c}{2} \cos \frac{c}{2}} \cos \frac{C}{2}$$

$$= \frac{2 \sin \frac{1}{2} c \cos \frac{1}{2}(a-b)}{2 \sin \frac{c}{2} \cos \frac{c}{2}} \cos \frac{C}{2}.$$

Hence

$$\sin\frac{A+B}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cos\frac{C}{2} \qquad [87]$$

Similarly,

$$\sin\frac{A-B}{2}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{c}{2}}\cos\frac{C}{2} \qquad [87']$$

$$\cos\frac{A+B}{2}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}\sin\frac{C}{2} \qquad [87'']$$

and

$$\cos\frac{A-B}{2}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{c}{2}}\sin\frac{C}{2} \qquad [87''']$$

These formulas are known as **Gauss's equations** or **Delambre's analogies.**

EXERCISE

Show in detail the derivation of the last three of the above formulas.

206. Rules of species for general triangles.—It will be seen that formulas [83] to [86] suffice for the solution of spherical triangles whenever sufficient parts are given to determine the triangle. From them certain rules of species may be deduced for application in case a particular formula leaves ambiguity.

From the law of cosines, formula [81],

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}.$$

The denominator of this fraction is always positive. In the numerator, $\cos a$ is greater arithmetically than $\cos b \cos c$ if a differs more from 90° than does either b or c. In that case, $\cos A$ has the same sign as does $\cos a$. Similarly from [82],

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C}.$$

Thus $\cos a$ has the same sign as $\cos A$ if A differs more from 90° than does either B or C. That is,

Any side (angle) of a spherical triangle is of the same species as its opposite angle (side) when that side (angle) differs more from 90° than does either of the two other sides (angles).

This rule will in general give unambiguous results for two sides of the triangle and the angles opposite them. A further rule is therefore necessary. This is deduced from the second of the four Napier analogies, according to which

$$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{c}{2}}{\tan \frac{1}{2}(a+b)}.$$

In this formula $\frac{1}{2}(A-B)$ and $\frac{c}{2}$ are always less than 90°, hence $\cos \frac{1}{2}(A-B)$ and $\tan \frac{c}{2}$ are always positive. Then $\cos \frac{1}{2}(A+B)$ and $\tan \frac{1}{2}(a+b)$ are of the same sign. That is, *half the sum of any two angles of a spherical triangle is of the same species as half the sum of the two opposite sides.*

207. Solution of general triangles.—In the solution of general spherical triangles we consider six cases according to the parts given.

Case I Three sides.
Case II Three angles.
Case III Two sides and their included angle.
Case IV Two angles and their included side.
Case V Two sides and the angle opposite one of them.
Case VI Two angles and the side opposite one of them.

Case I. If the three sides of a triangle are given, the angles may be found by using formulas [83] to [83″]. No ambiguity of sign can arise in this case. The solutions obtained can be checked readily by the sine law. The method of solution will be shown by an example.

Example. Find the angles of the triangle ABC, given:

$$a = 48° 55', b = 73° 21', c = 81° 38'$$

Solution: $$r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

$$\tan\frac{A}{2} = \frac{r}{\sin(s-a)},\ \tan\frac{B}{2} = \frac{r}{\sin(s-b)},\ \tan\frac{C}{2} = \frac{r}{\sin(s-c)}$$

$$\begin{aligned}
a &= 48^\circ\ 55' \\
b &= 73^\circ\ 21' \\
c &= \underline{81^\circ\ 38'} \\
2s &= 202^\circ\ 114' \\
s &= 101^\circ\ 57' \\
s-a &= 53^\circ\ \ 2' \\
s-b &= 28^\circ\ 36' \\
s-c &= \underline{20^\circ\ 19'} \\
s &= 101^\circ\ 57'
\end{aligned}
\qquad\qquad
\begin{aligned}
\log\sin(s-a) &= 9.90254 \\
\log\sin(s-b) &= 9.68006 \\
\log\sin(s-c) &= 9.54059 \\
\operatorname{colog}\sin s &= \underline{0.00952} \\
\log r^2 &= 9.13271 \\
\log r &= 9.56636 \\
\log\tan\tfrac{1}{2}A &= 9.66382 \\
\log\tan\tfrac{1}{2}B &= 9.88630 \\
\log\tan\tfrac{1}{2}C &= 0.02577
\end{aligned}$$

Therefore, $\frac{A}{2} = 24^\circ45'20'',\ \frac{B}{2} = 37^\circ35'2'',\ \frac{C}{2} = 46^\circ41'55''$

and $A = 49^\circ30'40'',\ B = \overset{75^\circ}{\cancel{74}}{}^\circ10'4'',\ C = 93^\circ23'50''.$

Check. $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\begin{aligned}
\log\sin a &= 9.87723 \\
\log\sin A &= \underline{9.88112} \\
&\ 9.99611
\end{aligned}
\qquad
\begin{aligned}
\log\sin b &= 9.98140 \\
\log\sin B &= \underline{9.98528} \\
&\ 9.99612
\end{aligned}
\qquad
\begin{aligned}
\log\sin c &= 9.99535 \\
\log\sin C &= \underline{9.99923} \\
&\ 9.99612
\end{aligned}$$

Case II. If the three angles are given, the procedure is very similar to that shown above, but on the basis of formulas **[84]** to **[84″]**; viz.,

$$R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}},$$

$$\tan\frac{a}{2} = R\cos(S-A),\ \tan\frac{b}{2} = R\cos(S-B),\ \tan\frac{c}{2} = R\cos(S-C).$$

EXERCISES

Find parts not given in the following:

1. $\begin{cases} a = 38^\circ\ 18' \\ b = 46^\circ\ 28' \\ c = 69^\circ\ 44' \end{cases}$

2. $\begin{cases} a = 112^\circ\ 25' \\ b = 96^\circ\ 20' \\ c = 28^\circ\ 35' \end{cases}$

3. $\begin{cases} A = 70^\circ\ 32' \\ B = 62^\circ\ 48' \\ C = 48^\circ\ 34' \end{cases}$

4. $\begin{cases} A = 68^\circ\ 10' \\ B = 76^\circ\ 18' \\ C = 154^\circ\ 36' \end{cases}$

5. $\begin{cases} a = 71^\circ\ 21' \\ b = 84^\circ\ 19' \\ c = 20^\circ\ 50' \end{cases}$

6. $\begin{cases} A = 36^\circ\ 30' \\ B = 118^\circ\ 17' \\ C = 78^\circ\ 35' \end{cases}$

7. $\begin{cases} A = 88^\circ\ 55' \\ B = 53^\circ\ 25' \\ C = 61^\circ\ 20' \end{cases}$

8. $\begin{cases} a = 116^\circ\ 30' \\ b = 36^\circ\ 24' \\ c = 82^\circ\ 6' \end{cases}$

9. $\begin{cases} A = 55^\circ\ 40' \\ B = 69^\circ\ 22' \\ C = 75^\circ\ 10' \end{cases}$

10. $\begin{cases} a = 126^\circ\ 35' \\ b = 75^\circ\ 5' \\ c = 79^\circ\ 40' \end{cases}$

11. $\begin{cases} A = 84^\circ\ 26' \\ B = 56^\circ\ 46' \\ C = 95^\circ\ 24' \end{cases}$

12. $\begin{cases} a = 33^\circ\ 18' \\ b = 46^\circ\ 25' \\ c = 59^\circ\ 41' \end{cases}$

Case III. When two sides and the included angle are given, the sum and difference of the two unknown angles can be found by formulas [85′] and [86′], and hence the values of the angles determined. Then by [85] the unknown side can be found and [86] furnishes a check for the computation. An illustrative example will be shown.

Example. Solve the spherical triangle ABC, given: $b = 118° \, 12'$, $c = 52° \, 41'$, $A = 78° \, 36'$.

Solution:

$$\tan \tfrac{1}{2}(B+C) = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \tfrac{1}{2} A.$$

$$\begin{aligned}
\log \cos \tfrac{1}{2}(b-c) &= 9.92477\\
\log \cot \tfrac{1}{2} A &= 0.08699\\
\text{colog} \cos \tfrac{1}{2}(b-c) &= \underline{1.09977}\\
\log \tan \tfrac{1}{2}(B-C) &= 1.11153\\
\tfrac{1}{2}(B-C) &= 85° \, 34' \, 37''
\end{aligned}$$

$$\tan \tfrac{1}{2}(B-C) = \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}(b+c)} \cot \tfrac{1}{2} A.$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(b-c) &= 9.73328\\
\log \cot \tfrac{1}{2} A &= 0.08699\\
\text{colog} \sin \tfrac{1}{2}(b-c) &= \underline{0.00137}\\
\log \tan \tfrac{1}{2}(B-C) &= 9.83716\\
\tfrac{1}{2}(B-C) &= 51° \, 13' \, 46''
\end{aligned}$$

$$\tan \frac{a}{2} = \frac{\sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B-C)} \tan \tfrac{1}{2}(b-c).$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(B+C) &= 9.99871\\
\log \tan \tfrac{1}{2}(b-c) &= 9.80850\\
\text{colog} \sin \tfrac{1}{2}(B-C) &= \underline{0.25752}\\
\log \tan \frac{a}{2} &= 0.06473\\
\frac{a}{2} &= 49° \, 15' \, 14''
\end{aligned}$$

$$\tan \frac{a}{2} = \frac{\cos \frac{1}{2}(B+C)}{\cos \frac{1}{2}(B-C)} \tan \tfrac{1}{2}(b+c)$$

$$\begin{aligned}
\log \cos \tfrac{1}{2}(B+C) &= 8.88716\\
\log \tan \tfrac{1}{2}(b+c) &= 1.09840\\
\text{colog} \cos \tfrac{1}{2}(B-C) &= \underline{0.07915}\\
\log \tan \frac{a}{2} &= 0.06471\\
a &= 98° \, 30' \, 28''
\end{aligned}$$

In this case, after the two angles have been found the law of sines may be used to find the remaining side, and a check is furnished by using the other parts in the law of sines. If this procedure is used, the rules for species given in §206 must be used to determine whether the final side is greater than, or less than, 90°.

Case IV. When two angles and the included side are given, the procedure is similar to that in case III. The formulas used are [85] and [86] to find the unknown sides, [85′] to find the remaining angle and then [86] may be used as a check. As before, the law of sines may be applied after the two unknown sides have been found.

EXERCISES

Solve the following triangles.

1. $\begin{cases} a = 44^\circ\ 50' \\ b = 29^\circ\ 10' \\ C = 75^\circ\ 16' \end{cases}$

2. $\begin{cases} A = 49^\circ\ 26' \\ B = 67^\circ\ 38' \\ c = 55^\circ\ 22' \end{cases}$

3. $\begin{cases} B = 78^\circ\ 31' \\ C = 62^\circ\ 15' \\ a = 50^\circ\ 32' \end{cases}$

4. $\begin{cases} A = 29^\circ\ 35' \\ C = 70^\circ\ 17' \\ b = 57^\circ\ 44' \end{cases}$

5. $\begin{cases} a = 12^\circ\ 26' \\ b = 32^\circ\ 44' \\ C = 98^\circ\ 30' \end{cases}$

6. $\begin{cases} b = 124^\circ\ 12' \\ c = 83^\circ\ 40' \\ A = 76^\circ\ 26' \end{cases}$

7. $\begin{cases} a = 96^\circ\ 34' \\ c = 63^\circ\ 46' \\ B = 71^\circ\ 54' \end{cases}$

8. $\begin{cases} B = 38^\circ\ 42' \\ C = 59^\circ\ 56' \\ a = 118^\circ\ 30' \end{cases}$

9. $\begin{cases} a = 117^\circ\ 58' \\ b = 82^\circ\ 12' \\ C = 57^\circ\ 36' \end{cases}$

Case V. When two sides and the angle opposite one of them are given, a second angle may be found by the law of sines.

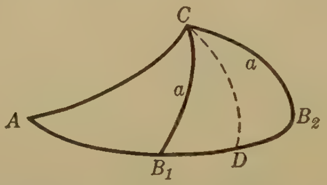
If A, b and a are given, then

$$\sin B = \frac{\sin b \sin \mathrm{A}}{\sin a}.$$

If $\sin a < \sin b \sin A$, then $\sin B > 1$ and no triangle exists for the given data.

If $\sin a = \sin b \sin A$, $\sin B = 1$ and the triangle is a right triangle.

If $\sin a > \sin b \sin A$, two supplementary values of B are possible. If $b < a$ then $B < A$, and if $b > a$ then $B > A$. If both values of b satisfy this, there are two triangles which satisfy the data of the problem; otherwise only one.



The figure shows the relation of two such triangles. Arc CD is perpendicular to arc AB. From Napier's rules, §193, $\sin CD = \sin b \sin A$.

When two sides and the angles opposite them are known, the remaining side may be found from formula [**85**], the remaining angle from [**85′**]. These two parts and the given opposite parts may then be used in the law of sines as a check on the work.

Example. Solve the spherical triangle ABC, given $a = 72^\circ\ 35'$, $b = 51^\circ\ 6'$, $B = 38^\circ\ 28'$.

Solution:

$$\sin A = \frac{\sin a \sin B}{\sin b}$$

$$\cot \frac{C}{2} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B)$$

$$\tan \frac{c}{2} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b)$$

$$\begin{aligned}\log\sin a &= 9.97962\\ \log\sin B &= 9.79383\\ \operatorname{colog}\sin b &= \underline{0.10888}\\ \log\sin A &= 9.88233\end{aligned}$$

$\therefore$ $A = 49° 41' 55''$
or $130° 18' 5''$
both of which are possible.

(a) Using the smaller value of A.

$$\begin{aligned}a+b &= 123° 41'\\ \frac{a+b}{2} &= 61° 50' 30''\\ a-b &= 21° 29'\\ \frac{a-b}{2} &= 10° 44' 30''\end{aligned}$$

$$\begin{aligned}A+B &= 88° 9' 55''\\ \frac{A+B}{2} &= 44° 4' 58''\\ A-B &= 11° 13' 55''\\ \frac{A-B}{2} &= 5° 36' 58''\end{aligned}$$

$$\begin{aligned}\log\sin \tfrac{1}{2}(a-b) &= 9.94530\\ \log\tan \tfrac{1}{2}(A-B) &= 8.99271\\ \operatorname{colog}\sin \tfrac{1}{2}(a-b) &= 0.72960\\ \log\cot\frac{C}{2} &= 9.66761\\ \frac{C}{2} &= 65° 3' 13''\\ C &= 130° 6' 26''\end{aligned}$$

$$\begin{aligned}\log\sin \tfrac{1}{2}(A-B) &= 9.84242\\ \log\tan \tfrac{1}{2}(a-b) &= 9.27808\\ \operatorname{colog}\sin \tfrac{1}{2}(A-B) &= 1.00938\\ \log\tan\frac{c}{2} &= 0.12988\\ \frac{c}{2} &= 53° 26' 33''\\ c &= 106° 53' 6''\end{aligned}$$

Check. $\dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$

$$\begin{aligned}\log\sin b &= 9.89112\\ \log\sin B &= \underline{9.79383}\\ &\ 0.09729\end{aligned}$$

$$\begin{aligned}\log\sin c &= 9.98087\\ \log\sin C &= \underline{9.88357}\\ &\ 0.09730\end{aligned}$$

(b) Using the larger value of A:

$$\begin{aligned}A+B &= 168° 46' 5''\\ \frac{A+B}{2} &= 84° 23' 3''\end{aligned}$$

$$\begin{aligned}A-B &= 91° 50' 5''\\ \frac{A-B}{2} &= 45° 55' 3''\end{aligned}$$

$$\begin{aligned}\log\sin \tfrac{1}{2}(a-b) &= 9.94530\\ \log\tan \tfrac{1}{2}(A-B) &= 0.01391\\ \operatorname{colog}\sin \tfrac{1}{2}(a-b) &= \underline{0.72960}\\ \log\cot\frac{C}{2} &= 0.68881\\ \frac{C}{2} &= 11° 34' 14''\\ C &= 23° 8' 28''\end{aligned}$$

$$\begin{aligned}\log\sin \tfrac{1}{2}(A-B) &= 9.99791\\ \log\tan \tfrac{1}{2}(a-b) &= 9.27808\\ \operatorname{colog}\sin \tfrac{1}{2}(A-B) &= \underline{0.14367}\\ \log\tan\frac{c}{2} &= 9.41966\\ \frac{c}{2} &= 14° 43' 32''\\ c &= 29° 27' 4''\end{aligned}$$

Check.

$$\begin{aligned}\log\sin c &= 9.69168\\ \log\sin C &= \underline{9.59439}\\ &\ 0.09729, \text{ as above.}\end{aligned}$$

Case VI. When two angles and the side opposite one of them are given, the side opposite the second given side may be found by use of the law of sines, and the remaining parts by [86] and [86′].

EXERCISES

Solve the following triangles.

Group A

1. $\begin{cases} a = 37^\circ\ 50' \\ b = 63^\circ\ 10' \\ B = 70^\circ\ 18' \end{cases}$

2. $\begin{cases} b = 75^\circ\ 28' \\ c = 83^\circ\ 21' \\ C = 40^\circ\ 33' \end{cases}$

3. $\begin{cases} a = 49^\circ\ 45' \\ A = 58^\circ\ 26' \\ C = 32^\circ\ 20' \end{cases}$

4. $\begin{cases} A = 78^\circ\ 12' \\ B = 63^\circ\ 48' \\ a = 50^\circ\ 27' \end{cases}$

5. $\begin{cases} a = 68^\circ\ 18' \\ c = 56^\circ\ 32' \\ C = 32^\circ\ 55' \end{cases}$

6. $\begin{cases} a = 82^\circ\ 6' \\ b = 69^\circ\ 15' \\ A = 118^\circ\ 20' \end{cases}$

7. $\begin{cases} B = 78^\circ\ 36' \\ C = 84^\circ\ 16' \\ c = 66^\circ\ 42' \end{cases}$

8. $\begin{cases} A = 96^\circ\ 47' \\ C = 105^\circ\ 52' \\ c = 72^\circ\ 11' \end{cases}$

9. $\begin{cases} a = 65^\circ\ 26' \\ c = 53^\circ\ 14' \\ A = 85^\circ\ 28' \end{cases}$

10. $\begin{cases} b = 50^\circ\ 20' \\ c = 58^\circ\ 46' \\ B = 98^\circ\ 10' \end{cases}$

11. $\begin{cases} A = 70^\circ\ 23' \\ B = 88^\circ\ 13' \\ b = 81^\circ\ 34' \end{cases}$

12. $\begin{cases} B = 62^\circ\ 37' \\ C = 73^\circ\ 21' \\ b = 49^\circ\ 41' \end{cases}$

Group B

1. $\begin{cases} a = 78^\circ\ 20' \\ b = 59^\circ\ 18' \\ A = 68^\circ\ 16' \end{cases}$

2. $\begin{cases} B = 87^\circ\ 30' \\ C = 71^\circ\ 22' \\ b = 75^\circ\ 45' \end{cases}$

3. $\begin{cases} a = 71^\circ\ 27' \\ c = 47^\circ\ 34' \\ A = 62^\circ\ 12' \end{cases}$

4. $\begin{cases} b = 82^\circ\ 42' \\ c = 110^\circ\ 33' \\ C = 85^\circ\ 50' \end{cases}$

5. $\begin{cases} A = 51^\circ\ 28' \\ B = 67^\circ\ 32' \\ b = 70^\circ\ 40' \end{cases}$

6. $\begin{cases} A = 103^\circ\ 30' \\ C = 58^\circ\ 8' \\ c = 47^\circ\ 13' \end{cases}$

7. $\begin{cases} a = 70^\circ\ 32' \\ b = 72^\circ\ 16' \\ B = 81^\circ\ 7' \end{cases}$

8. $\begin{cases} A = 85^\circ\ 10' \\ C = 82^\circ\ 19' \\ a = 56^\circ\ 35' \end{cases}$

9. $\begin{cases} B = 60^\circ\ 50' \\ C = 48^\circ\ 27' \\ c = 27^\circ\ 48' \end{cases}$

10. $\begin{cases} a = 74^\circ\ 14' \\ c = 82^\circ\ 17' \\ C = 107^\circ\ 20' \end{cases}$

11. $\begin{cases} A = 93^\circ\ 11' \\ B = 81^\circ\ 13' \\ a = 73^\circ\ 40' \end{cases}$

12. $\begin{cases} b = 105^\circ\ 25' \\ c = 116^\circ\ 32' \\ B = 48^\circ\ 44' \end{cases}$

HISTORICAL SKETCH

CHAPTER XV

HISTORICAL SKETCH

The principal sources used in compiling this historical sketch are:

(1) Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, fourth edition, 1906.

(2) Florian Cajori, *A History of Mathematics*, second edition, 1924.

(3) W. W. Rouse Ball, *A Short Account of the History of Mathematics*, 1893.

(4) Heinrich Wieleitner, *Geschichte der Mathematik*, (Sammlung Schubert, LXIV) 1921.

The historical development of any science, or even of culture as a whole, often appears piecemeal and illogical. Parts which later are closely connected by apparently inevitable logical relations appear at great intervals of time and in widely separated geographical regions. Nor do these parts necessarily appear in the order of their simplicity, when simplicity is measured by the comparative difficulty of understanding, once the discovery has been made. A submerged continent is rising out of the sea: an island, a chain of islands, and then a whole region appear; later other islands and regions arise in distant places, and in time these are all connected as the continent emerges. The early spots give little promise of the final unit, and their significance is apparent only when the event has been consummated.

But the figure is not wholly fortunate. The beginnings of a science are often developed as parts of other and more general sciences. The epoch-making events in the history of science are the discovery of new methods of attack and new general theories and points of view, quite as much as the discovery of new facts. Until these appear, investigations by means of the older methods are frequently pushed to their very limit. Great intellects achieve surprising results, while that which at a later time appears simple and obvious remains unnoticed. The history of the development of trigonometry furnished numerous instances. Many of its more difficult theorems were stated and proved long before the invention of the system of numerals which we now study in our grade schools.

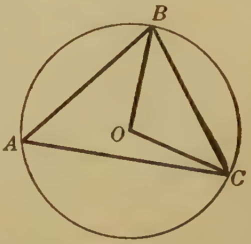
GREEK TRIGONOMETRY

Elementary trigonometry is essentially a combination of geometry and methods of computation, with a rather slight admixture of algebra. The geometry derives from the Greeks, the system of numerals and the algebra from the Hindus and Arabs, and some of the methods and means of computation, notably logarithms, from more recent European times. It is nevertheless true that trigonometry originated with the Greeks. To understand what they contributed to this science, and what they failed to contribute, we must know what was the main field of Greek work in mathematics, and also something about the limitations under which they labored.

Beyond any reasonable comparison the greatest mathematical work of the Greeks was in geometry. Using methods with which we are in part familiar from our study in the secondary schools, they pushed their investigations farther than any subsequent workers who were limited as they were. The whole geometric basis for trigonometry was perfected by them. But they were cramped by certain shortcomings which made it impossible for them to develop the science as we know it today, though to us it is vastly simpler than the results in pure geometry obtained by such transcendent geniuses as Archimedes and Apollonius.

The three theorems (8), (9), (10), on page 142, which contain the substance of the cosine formula, $a^2 = b^2 + c^2 - 2bc\cos A$, are found in Euclid's *Elements*, but there is a vast gap between these theorems and the complete formula as we understand it. Greek geometry made little use of numbers. With them, for instance, the product of two lines (or the square of a line) was a rectangle, while with us it is the product of two numbers representing the lengths of the lines. With the Greeks the idea of number was essentially limited to positive (or unsigned) integers. Fractions caused them great trouble and they were wholly ignorant of our decimal notation. In Euclid's *Elements* a complete theory of proportion is developed on a purely geometric basis with no reference to numbers. Apparently it did not occur to them that the same relations exist between numbers. With such limitations it is evident that they could not develop in its entirety what is now to a large extent a computational science. But they did a great deal.

The sine formula, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, which is the fundamental theorem in the computational part of trigonometry, was not stated explicitly, but was contained implicitly in other theorems. If the Greeks had stated this theorem it would run in substance thus: *The sides of a triangle are in the same ratio as the chords of twice the opposite angles.* This is at once evident from the adjoining figure. The side BC is the chord of the angle BOC (the Greeks thought of it as the chord of the *arc* BC) which is twice the angle A. Tables of chords which are thus seen to be essentially equivalent to tables of natural sines were constructed and used. Hipparchus, the most eminent of Greek astronomers, who lived in the second century before Christ, is generally regarded as the creator of the subject of trigonometry. He is known to have constructed a table of chords, but we do not know what methods he used.

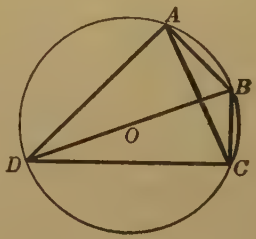


Ptolemy of Alexandria, who flourished about the middle of the second century A.D., collected and organized the then existing science of astronomy in his great work, the *Almagest.* It is not known how much of this work was original with Ptolemy, though it is certain that a large part of it was due to Hipparchus and other investigators.

The *Almagest* is divided into thirteen books. In the first book he gives (along with other matter) an account of trigonometry, including a table of chords and the method used in computing them. Use was made of the following fundamental theorem, the so-called Ptolemaic theorem:

If a quadrilateral is inscribed in a circle, then the product of its diagonals is equal to the sum of the products of the two pairs of opposite sides.

Thus in the figure $AC \times DB = AB \times DC + BC \times AD$. By means of this theorem the chords of the sum or of the difference of two arcs may be found when the chords of the arcs themselves are known, as may also the chord of half an arc or twice an arc.



Thus suppose the chords AB and BC are known, and it is required to find the chord AC.

Let DB be a diameter of known length.

Then $$AC = \frac{AB \times DC + BC \times AD}{DB}.$$

Since DCB and DAB are right angles,

$$DC = \sqrt{DB^2 - BC^2} \text{ and } AD = \sqrt{DB^2 - AB^2},$$

and $$AC = \frac{AB\sqrt{DB^2 - BC^2} + BC\sqrt{DB^2 - AB^2}}{DB}$$

By similar constructions, the chord of the difference of two arcs and the chord of half an arc are found.

By beginning with arcs of known chords such as 60° and 72°, through successive halving, subtracting, and adding of arcs, Ptolemy obtained the chords of a series of arcs such as 30°, 15°, 7½°, 12°, 6°, 3°, 1½°, etc. Ptolemy gives the chords of arcs at 30 minute intervals. To do this it was necessary to find the chord of one degree. This was done by means of a method of approximation based on an ingenious theorem.

The Greeks from Hipparchus on divided the circle into 360°, one degree into 60′, and one minute into 60″. That is, they used the sexagesimal system in dealing with arcs and angles, while their general number system was decimal, as ours is. It is fairly certain that this system was borrowed from the Babylonians who used it in general. It is not known, however, why the Babylonians came to divide the circle into 360 parts, though many conjectures have been made. Perhaps the most plausible of these conjectures is that one of the equal angles of an equilateral triangle was regarded as a unit angle. Using sexagesimal division then gives 60° in such an angle, or 360° in the whole circle.

The form which the Greeks used in constructing their tables is interesting. The radius of the circle was divided into 60 equal parts, and the chords were given in terms of one of these parts as a unit. To get closer approximations this unit was divided into 60 parts, called minute parts, and one of these minute parts was again divided into 60 parts, called second minute parts. So a chord was given as so many parts, so many minute parts, and so many second minute parts. Thus the chord of the arc of 20°30′ is given as 21°21′12″. This means that this chord is $\frac{21}{60} + \frac{21}{3600} + \frac{12}{216000}$ of the radius. It is interesting that the number of each of these parts was given in the decimal number system, but of course in a system of numerals very different from ours—a system which was much like that of the Romans, though somewhat clumsier. When translated into Latin these smaller parts were written *partes minutæ* and *partes minutæ secundæ.* Hence we have our *minutes* and *seconds.*

A table of proportional parts was given for interpolation between each successive half-degree of arc, the assumption being that within these narrow limits the chords change in proportion to the arc. This is precisely the assumption which we have made for the much narrower limit of one minute of arc. As in the main table, so also in the table of proportional parts, the sexagesimal system is used both for arcs and for chords.

The value of π used by Ptolemy is $3\ 8\ 30'' = 3 + \frac{8}{60} + \frac{30}{3600} =$ 3.141666, the radius being unity.

Heron of Alexandria, who lived in the latter part of the second century B.C., discovered the formula $A = \sqrt{s(s-a)\ (s-b)\ (s-c)}$ giving the area of the triangle (see page 82). He illustrated the use of this formula by finding the area of a triangle whose sides are 13, 14, 15. In finding this formula, Heron departed from the ways of the Greeks. Ball says of him:

> Hero [Heron] did nothing to extend a knowledge of abstract mathematics; . . . he was interested in science only on account of its practical applications, and so long as his results were true he cared nothing for the logical accuracy of the process by which he arrived at them. Thus in finding the area of a triangle he took the square root of the product of four lines. The classical Greek geometricians permitted the use of the square or the cube of a line [or the product of any two or three lines] because these could be represented geometrically, but a figure of four dimensions is inconceivable, and certainly they would have rejected a proof involving such a conception.

It is to be noted that the astronomers and not the mathematicians developed trigonometry in Greece. Hipparchus and Ptolemy were astronomers. For this reason spherical trigonometry was developed more fully than plane.

This account of Greek contribution to trigonometry may be closed appropriately with the following quotation from Cantor:

> The ancients did not make use of trigonometry either in their theory of plane areas or in their practice in surveying, with the single exception of the numerical formulas of Heron for the areas of regular polygons. This fact

may appear surprising at first sight, but an explanation does not appear difficult.

Trigonometric expressions are not thinkable as a basis upon which to build other species of magnitudes so long as there is not available an organized system of mathematical symbols. In the absence of such a symbolism, trigonometric expressions expressed in tables of lines give only approximately correct results. But the scientific geometrician was not inclined to be satisfied with mere approximations no matter how close these might be. The practical surveyor on the other hand was disinclined to master the science which was indispensable in order to employ trigonometric computation. So it came about that they left the disrespected or feared methods of trigonometry to the astronomer who, being less highly critical than the geometrician and intellectually less indolent than the surveyor, made cheerful use of their approximate results.

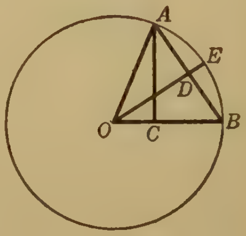
Hindu and Arab Contributions to Trigonometry

The earliest known Hindu writings containing elements of trigonometry date from the fifth or sixth century of our era, and the latest Arab contributions were made about the year 1250. From intrinsic evidence in their writings, as well as from general probability, it is safe to assume that these Eastern peoples had learned from the Greeks. But the science was largely transformed, if not greatly extended, in their hands. As the Greeks were masters in dealing with purely geometric problems, so the particular genius of the Hindus and Arabs was in dealing with numbers and in computation, including the more general computation of algebra. Even Greek algebra was geometric in character. The unknown was a line if of the first degree, and its square represented an area. With the Hindus and the Arabs these were pure numbers, as they are with us. The system of numerals which we now use had been invented in India, the essential element of this system being the principle of place value. As a consequence trigonometric functions came to be used as symbols representing numerical values and practical computation was greatly facilitated.

One of the most far-reaching innovations made by the Hindus was the replacing of the chord of an arc by half the chord of double the arc—that is by its sine as we now have it. Their method of computing also differed somewhat from that of the Greeks. Like the Greeks, and before them the Babylonians, the Hindus divided the circle into 360 degrees and these into minutes and seconds. But here a difference arises. The 360 degrees in the circle equal 21,600 minutes. While the Greeks divided the radius into 60 equal parts, the Hindus raised the question as to the number of these minutes of arc which was equal in length to the radius. Using 3.1416 as the value of π, they found this number to be 3,438 when taken to the nearest unit. That is, using the length of a minute of arc as a unit they wrote $r = 3438$. Aryabhata (born 476 A.D.) proceeds as follows to find $\sin \frac{\alpha}{2}$, when $\sin \alpha$ is known. In the figure let arc $AB = \alpha$ and $AE = \frac{\alpha}{2}$.

Then (using the Hindu definition of sine)

$\sin \alpha = CA$ and $\sin \frac{\alpha}{2} = DA$.



But $AB^2 = AC^2 + CB^2 = (r + OC)\ (r - OC) + (r - OC)^2 = 2r^2 - 2r \cdot OC$.

Hence $\sin^2 \frac{\alpha}{2} = \frac{r^2 - r \cdot OC}{2} = \frac{r}{2}\ (r - OC)$,

and

$$\sin \frac{\alpha}{2} = \sqrt{\frac{r}{2}\,(r - OC)}.$$

If OC is denoted by $\cos \alpha$ and $r = 3438$, we have

$$\sin \frac{\alpha}{2} = \sqrt{1719\ (3438 - \cos \alpha)}$$

But it was known that $\sin^2 + \cos^2 = r^2$ and cos could be computed.

With this formula [or its equivalent], says Cajori, and with the sines of 90°, 60°, 45°, and 30° as starting-points, they reckoned the sines of half the angles [arcs?], thus obtaining the sines of 22°30, 15°, 11°15, 7°30, 3°45. They now figured out the sines of the complements of these angles [arcs?], namely, the sines of 86°15, 82°30, 78°45, 75°, 67°30; then they calculated the sines of half these angles [arcs?]; then of their complements; then, again, of half their complements; and so on. By this very simple process they got the sines of angles [arcs?] at intervals of 3°45.

Values of sin 15°, sin 7°30, sin 3°45 contained in the earliest known Hindu writings on mathematics were probably obtained by this method.

But [to quote from Cantor], Bhaskara [born 1114] was not satisfied with these tables. He computed sines and cosines as fractions of the radius: $\sin 225 = \frac{100}{1529}$, $\cos 225 = \frac{466}{467}$; $\sin 1° = \frac{10}{573}$, $\cos 1° = \frac{6568}{6569}$, when each fraction is a fraction of the radius.

As a final estimate of Hindu contribution to trigonometry Cantor says:

> Trigonometry as the computation of parts of any triangle by means of functions of angles did not exist for the Hindus. By them all problems on triangles were reduced to problems on right triangles, and then they were able to solve the problems that appeared.
>
> As essential progress of trigonometry in India there remains therefore the table of sines. Chords were permanently displaced by the half chords [sines].
>
> It is safe to say that this exceptionally fortunate throw of the Hindus was achieved quite by accident, since they did not understand the far-reaching importance of this step. This was reserved for their followers, the Arabs.

There is not space to sketch even in outline the story of the rise of the Arabs. When the eastern branch of the Mohammedans had organized their capital in Bagdad, their rulers began to seek the aid of scholars among the many peoples they had subdued or rather absorbed into their own faith. A period of translation ensued. Natives of Syria, Persia, Greece, and India, residing at Bagdad, translated literary and scientific treasures from their own vernaculars. So it came about that before the end of the ninth century Arab scholars had at their disposal practically all the science that had existed up to that time. Like the Greeks and the Hindus, they accepted the sexagesimal division of the angle, but they used the Hindu table of sines instead of the chords of the Greeks.

As was the case with the Greeks and the Hindus, Arab trigonometry, with one notable exception, was developed as a part of astronomy; but important advances were made. All the trigonometric functions that are now used were introduced. Al-Battani (877–929) used arcs with unit radius and in reality regarded the functions as ratios. He was the first to prepare a table of cotangents and he probably knew the law of sines.

Abu'l Wefa (940–98) introduced all the present functions and constructed tables of tangents and cotangents. In considering the shadow triangle he used the secant and cosecant.

> He invented a method for computing a table of sines which gives the sine of half a degree to nine decimal places. [Cajori.]

The history of the origin of the word *sine* is interesting.

> The work of Al-Battani, De scientia stellarum, was translated into Latin by Plato Tiburtinus, in the twelfth century. Out of this translation sprang

the word "sinus," as the name of a trigonometric function. The Arabic word for "sine," jiba was derived from the Sanskrit jiva, and resembled the Arabic word jaib, meaning an indentation or gulf. Hence the Latin "sinus."

It remains to mention Nasir-Eddin (1200–74), who more than any of his predecessors dissociated trigonometry from astronomy and who seems to have been the first to state explicitly the sine formula for plane triangles in practically its present form. To quote Cajori:

> During the supremacy of Hulagu, lived Nasir-Eddin, a man of broad culture and an able astronomer. He persuaded Hulagu to build him and his associates a large observatory at Maraga. Treatises on algebra, geometry, arithmetic, and a translation of Euclid's *Elements*, were prepared by him. He for the first time elaborated trigonometry independently of astronomy and to such great perfection that, had his work been known, Europeans of the fifteenth century might have spared their labors.

It must be admitted that the contributions of the Arabs were in no sense revolutionary. They elaborated and amplified, but they were commentators rather than creators of new theories and methods. Ball expresses himself as follows:

> The general impression left on my mind is that the Arabs were quick to appreciate the work of others—notably the Greek masters and of the Hindoo mathematicians—but, like the ancient Chinese and Egyptians, they were unable to systematically develop a subject to any considerable extent. Their schools may be taken to have lasted in all for about 650 years, and if the work produced be compared with that of Greek or modern European writers it is as a whole second-rate both in quantity and quality.

Trigonometry in Modern Europe

In what respect did the trigonometry of Nasir-Eddin differ from the elementary parts of the modern science? This question is best answered by enumerating those parts of the subject contained in a text such as the present which were not in the Arab treatise.

There was still wanting the algebraic notation which we now have. None of the present symbols of operation was in existence. The smoothness of modern algebraic operations, due very largely to a perfected notation, was impossible. There was also lacking an effective set of symbols to represent the trigonometric functions. For these reasons the derivation of formulas was much more difficult than at present.

Logarithms had not been invented and all computations were made by the ordinary processes of elementary arithmetic. Formulas specially adapted to logarithmic computation, such as the tangent formula, were of course wanting.

There was no system of coördinates such as we now use in giving general definitions of the trigonometric functions, the existence of negative numbers was scarcely realized, and consequently general definitions of the functions were impossible. Another three hundred years passed before the formula $a^2 = b^2 + c^2 - 2bc \cos A$ could be stated with its present complete meaning, though the facts which it represents were formulated into rigorously proved theorems by the Greeks.

The periodicity of the functions, as considered in Chapter IX, was obviously not recognized because of the restrictions on the definitions. Inverse functions in the more general sense and the multiplicity of solutions of trigonometric equations, the representation of the functions by means of infinite series, and such special developments as De Moivre's theorem were wholly lacking.

While a general decimal system of notation was in use, fractions continued to be written in the sexagesimal base until about 1620. Thus Leonardo of Pisa in the year 1225 wrote 1.3688081075 as $1.22' \; 7'' \; 42''' \; 33''''4^{\mathrm{V}} \; 40^{\mathrm{VI}}$.

The trigonometry of Nasir-Eddin would enable a competent mathematician to solve all problems in this text involving mensuration, but the work would be extremely laborious. Tables of natural trigonometric functions could be constructed, but the process was much more tedious than it is when series are available.

The use of trigonometry in higher mathematics and its applications in science were of course wanting—the mathematics and the science were not in existence.

The story of the further development of algebra and of the invention of a system of coördinates will not be considered in this chapter. We will confine ourselves to the remaining elements which are now parts of an introductory course such as the present.

The Arabs guarded their scientific treasures jealously, especially from the Christians. But little by little they were "smuggled" into Christian Europe. There were adventurous spirits, such as Adelhard of Bath, who about 1120 attended lectures at Cordova under the guise of a Mohammedan student; learned Jewish rabbis left Spain for Christian Europe; there was considerable commercial intercourse between Italy and the Arabs both west and east. So by a gradual process the principal manuscripts that had been translated into Arabic, or which had been created by the Arabs themselves, came into the hands of European scholars. (The work of Nasir-Eddin seems to be an exception.) Upon the basis of this body of knowledge the contributions of Europe were made.

The earliest European writings on trigonometry are due to Thomas Bradwardine (1290–1349), Archbishop of Canterbury. He speaks of tangent (umbra versa) and cotangent (umbra recta). Regiomontanus (John Mueller) (1435–76) studied trigonometry at Vienna. He translated the *Almagest* and constructed a table of tangents. Immediately after the time of Regiomontanus, the Germans calculated many tables, some of them very extensive. One of these calculators of tables, generally called Raetius, was the first to construct the right triangle and make use of it in defining the functions. That is, he made them depend upon the *angle* and not upon the arc, as had been done up to this time.

Logarithms were invented by John Napier, Baron of Merchiston, Scotland, and a table was published in 1614. Napier's logarithms were made to depend upon a correspondence between an arithmetic and a geometric series, and were in no way related to exponents. The latter came into use many years later. It was only in the time of Euler, about 1750, that logarithms came to be regarded as exponents.

Napier's tables differed in many respects from those we now use. His friend Briggs constructed a table using the base 10, the general plan for the Briggsian table being the joint product of Napier and Briggs. Napier died in 1617, but Briggs in 1624 published his *Arithmetica logarithmica* containing 14-place logarithms of numbers from 1 to 20,000 and from 90,000 to 100,000. The gap was filled by Adrian Vlacq, who was born in Holland and lived in London and Paris. In 1628 he published the complete table up to 100,000, 70,000 of which were computed by himself. Says Cajori:

> Briggs and Vlacq published four fundamental works, the results of which have not been superseded by any subsequent calculations until very recently.
>
> The word "characteristic", as used in logarithms, first occurs in Briggs' *Arithmetica logarithmica*, 1624; the word "mantissa" was introduced by John Wallis, 1693.
>
> Briggs divided a degree into 100 parts, as was done also by N. Roe in 1633, W. Oughtred in 1657, and John Newton in 1658, but owing to the publication by Vlacq of trigonometrical tables constructed on the old sexagesimal division, Briggs' innovation did not prevail.

Edmund Gunter of London found the logarithmic sines and tangents for every minute to seven places. He was the inventor of the words cosine and cotangent.

Abraham de Moivre (1667–1754), the author of the theorem known by his name (page 130), was of French descent and lived in London, where he enjoyed the high respects of the mathematicians of his time.

> Newton himself, in the later years of his life, used to reply to inquiries respecting mathematics, even respecting his Principia: "Go to Mr. De Moivre: he knows these things better than I do." . . . Shortly before his death he declared it necessary for him to sleep ten or twenty minutes longer every day. The day after he had reached the total of over twenty-three hours, he slept exactly twenty-four hours and then passed away in his sleep. [Cajori.]

Oughtred invented the slide rule (1621), Daniel Bernouli (1729) used AS to represent arcsine ($\sin^{-1}$), and Leonard Euler (1736) used At for arctangent. The theorem now known as Mollweide's theorem (Ex. 19, page 167) was first published by Simpson (1748), and in part by Newton in his *Universal Arithmetic* (1707). Mollweide rediscovered this theorem a century after Newton's original discovery.

The exponential forms given in Chapter XII, and infinite series used in computing trigonometric functions, were by-products of extensive investigations in many fields of mathematics in the eighteenth century. The series resulted from the work of Taylor and Maclaurin and the exponential forms are due mainly to Euler. The geometric representation of the imaginary, $\sqrt{-1}$, is due to Wessel (1797), to Argand (1806), and to Gauss, who published a fuller exposition in 1831. The systematic use of A, B, C, a, b, c to represent the angles and sides of a triangle was first adopted by Euler, but was used at times in earlier writings. The periodicity of the trigonometric functions was known by Cotes (1682–1716), but was developed later by Euler.

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FIVE-PLACE TABLES

LOGARITHMS
LOGARITHMIC FUNCTIONS
NATURAL FUNCTIONS

COMPILED BY
N. J. LENNES
AND
A. S. MERRILL
University of Montana

HARPER & BROTHERS PUBLISHERS
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F–G

PREFACE

Another set of five-place tables of logarithms and trigonometric functions requires justification. Such justification, we believe, is to be found in the exceptionally convenient form of these tables, in the attractiveness and general excellence of the typography, and in a degree of accuracy which is known to be almost, if not entirely, absolute.

The type is of good size and body, all numerals are of the same height, and the general result is such as to make continued use particularly easy. In every case, that part of a number which is to be prefixed to the numerals given is found by looking above in the same column. This avoids the use of asterisks and the looking across the page each time a logarithm is to be found. The result is a real advantage both in elementary instruction and in practical use.

An accurate set of five-place tables is perhaps not as common as one might expect. The set from which the original copy for these tables was made had several scores of errors. Our random selection for this purpose may have been unfortunate.

The manuscript of logarithms of numbers and of logarithmic functions was corrected by reading against Bremiker's edition of Vega's seven-place tables. No table of natural functions of known reliability being available, this part of the table was recomputed completely. For this purpose the logarithms of these functions were found in Schrön's seven-place table and the numbers finally determined by using the tables of the U. S. Coast and Geodetic Survey. Schrön's table, using a system indicating plus or minus after the last figure, gives each logarithm to within 5 in the eighth place. This left no doubt as to the figure in the fifth place of the numbers except in about a half-dozen cases where the correct value is almost exactly half way between two consecutive five-place numbers. It is possible that in some of these cases accuracy might be improved by changing the figure in the fifth place, but we know that the maximum improvement that is thus possible is one unit in the seventh place.

The cast proof of the tables of logarithms and logarithmic functions was read by separate teams against the Schrön and the Bruhn tables, this reading disclosing in all five errors. In every case the final figure was decided upon by using the Schrön table with its final plus and minus marking. The logarithms of numbers were also read against the ten-place table of the U. S. Coast and Geodetic Survey. No discrepancies between these tables were detected except one error in a heading in the Bruhn table.

The cast proof of the table of natural functions was read against Loomis' six-place table, which however fails to decide the last figure in our table, in about seven hundred cases. These were recomputed. In this reading and recomputing of the cast table of natural functions one error was found. Also one error was found in the Loomis table which gives cot 32° 24′ = 1.57555 instead of 1.57575. If the very high degree of accuracy of the Loomis table had been known to us we could have spared ourselves much computing, though it would still remain to settle cases of doubtful last figures.

The tables of differences were computed directly from the columns of logarithms and this provided an additional check against gross errors in the tables of logarithms and logarithmic functions.

As an introduction, when the tables are bound separately, the chapter on logarithms in our trigonometry has been used. This affords the most convenient arrangement for the initial study of the use of the tables that has come to our attention. For instance, on page XII (in *Plane Trigonometry*, p. 34) there is the first study of finding logarithms of numbers and this refers solely to the table on the page opposite. Finding antilogarithms is introduced on page XIV (p. 36) and the page of table opposite contains all of the logarithms that are used. A similar arrangement is made on pages XXII and XXIII (pp. 44 and 45) for learning to use the table of logarithmic functions. Appropriate exercises are given at each step so that the table may be used readily in the class room for practice in the use of logarithms without arranging additional material.

N. J. Lennes
A. S. Merrill

LOGARITHMS OF NUMBERS

Numbers 100–150 Logs 00000–17869

N	0	1	2	3	4	5	6	7	8	9	P.P.			
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389		**43**	**42**	**41**
101	432	475	518	561	604	647	689	732	775̄	817	1	4.3	4.2	4.1
102	860	903	945	988	01 030	01 072	01 115̄	01 157	01 199	01 242	2	8.6	8.4	8.2
103	01 284	01 326	01 368	01 410	452	494	536	578	620	662	3	12.9	12.6	12.3
104	703	745	787	828	870	912	953	995̄	02 036	02 078	4	17.2	16.8	16.4
105	02 119	02 160	02 202	02 243	02 284	02 325	02 366	02 407	449	490	5	21.5	21.0	20.5
106	531	572	612	653	694	735̄	776	816	857	898	6	25.8	25.2	24.6
107	938	979	03 019	03 060	03 100	03 141	03 181	03 222	03 262	03 302	7	30.1	29.4	28.7
108	03 342	03 383	423	463	503	543	583	623	663	703	8	34.4	33.6	32.8
109	743	782	822	862	902	941	981	04 021	04 060	04 100	9	38.7	37.8	36.9
110	04 139	04 179	04 218	04 258	04 297	04 336	04 376	04 415̄	04 454	04 493		**40**	**39**	**38**
111	532	571	610	650	689	727	766	805	844	883	1	4.0	3.9	3.8
112	922	961	999	05 038	05 077	05 115	05 154	05 192	05 231	05 269	2	8.0	7.8	7.6
113	05 308	05 346	05 385̄	423	461	500	538	576	614	652	3	12.0	11.7	11.4
114	690	729	767	805̄	843	881	918	956	994	06 032	4	16.0	15.6	15.2
115	06 070	06 108	06 145	06 183	06 221	06 258	06 296	06 333	06 371	408	5	20.0	19.5	19.0
116	446	483	521	558	595	633	670	707	744	781	6	24.0	23.4	22.8
117	819	856	893	930	967	07 004	07 041	07 078	07 115̄	07 151	7	28.0	27.3	26.6
118	07 188	07 225̄	07 262	07 298	07 335	372	408	445	482	518	8	32.0	31.2	30.4
119	555̄	591	628	664	700	737	773	809	846	882	9	36.0	35.1	34.2
120	07 918	07 954	07 990	08 027	08 063	08 099	08 135̄	08 171	08 207	08 243		**37**	**36**	**35**
121	08 279	08 314	08 350	386	422	458	493	529	565̄	600	1	3.7	3.6	3.5
122	636	672	707	743	778	814	849	884	920	955	2	7.4	7.2	7.0
123	991	09 026	09 061	09 096	09 132	09 167	09 202	09 237	09 272	09 307	3	11.1	10.8	10.5
124	09 342	377	412	447	482	517	552	587	621	656	4	14.8	14.4	14.0
125	691	726	760	795	830	864	899	934	968	10 003	5	18.5	18.0	17.5
126	10 037	10 072	10 106	10 140	10 175	10 209	10 243	10 278	10 312	346	6	22.2	21.6	21.0
127	380	415̄	449	483	517	551	585	619	653	687	7	25.9	25.2	24.5
128	721	755̄	789	823	857	890	924	958	992	11 025	8	29.6	28.8	28.0
129	11 059	11 093	11 126	11 160	11 193	11 227	11 261	11 294	11 327	361	9	33.3	32.4	31.5
130	11 394	11 428	11 461	11 494	11 528	11 561	11 594	11 628	11 661	11 694		**34**	**33**	**32**
131	727	760	793	826	860	893	926	959	992	12 024	1	3.4	3.3	3.2
132	12 057	12 090	12 123	12 156	12 189	12 222	12 254	12 287	12 320	352	2	6.8	6.6	6.4
133	385	418	450	483	516	548	581	613	646	678	3	10.2	9.9	9.6
134	710	743	775	808	840	872	905̄	937	969	13 001	4	13.6	13.2	12.8
135	13 033	13 066	13 098	13 130	13 162	13 194	13 226	13 258	13 290	322	5	17.0	16.5	16.0
136	354	386	418	450	481	513	545	577	609	640	6	20.4	19.8	19.2
137	672	704	735	767	799	830	862	893	925̄	956	7	23.8	23.1	22.4
138	988	14 019	14 051	14 082	14 114	14 145̄	14 176	14 208	14 239	14 270	8	27.2	26.4	25.6
139	14 301	333	364	395	426	457	489	520	551	582	9	30.6	29.7	28.8
140	14 613	14 644	14 675̄	14 706	14 737	14 768	14 799	14 829	14 860	14 891		**31**	**30**	**29**
141	922	953	983	15 014	15 045̄	15 076	15 106	15 137	15 168	15 198	1	3.1	3.0	2.9
142	15 229	15 259	15 290	320	351	381	412	442	473	503	2	6.2	6.0	5.8
143	534	564	594	625̄	655̄	685	715	746	776	806	3	9.3	9.0	8.7
144	836	866	897	927	957	987	16 017	16 047	16 077	16 107	4	12.4	12.0	11.6
145	16 137	16 167	16 197	16 227	16 256	16 286	316	346	376	406	5	15.5	15.0	14.5
146	435	465	495̄	524	554	584	613	643	673	702	6	18.6	18.0	17.4
147	732	761	791	820	850	879	909	938	967	997	7	21.7	21.0	20.3
148	17 026	17 056	17 085̄	17 114	17 143	17 173	17 202	17 231	17 260	17 289	8	24.8	24.0	23.2
149	319	348	377	406	435	464	493	522	551	580	9	27.9	27.0	26.1
150	17 609	17 638	17 667	17 696	17 725̄	17 754	17 782	17 811	17 840	17 869				
N	0	1	2	3	4	5	6	7	8	9				

N	0	1	2	3	4	5	6	7	8	9	P.P.		
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869		29	28
151	898	926	955	984	18 013	18 041	18 070	18 099	18 127	18 156	1	2.9	2.8
152	18 184	18 213	18 241	18 270	298	327	355	384	412	441	2	5.8	5.6
153	469	498	526	554	583	611	639	667	696	724	3	8.7	8.4
154	752	780	808	837	865	893	921	949	977	19 005	4	11.6	11.2
155	19 033	19 061	19 089	19 117	19 145	19 173	19 201	19 229	19 257	285	5	14.5	14.0
156	312	340	368	396	424	451	479	507	535	562	6	17.4	16.8
157	590	618	645	673	700	728	756	783	811	838	7	20.3	19.6
158	866	893	921	948	976	20 003	20 030	20 058	20 085	20 112	8	23.2	22.4
159	20 140	20 167	20 194	20 222	20 249	276	303	330	358	385	9	26.1	25.2
160	20 412	20 439	20 466	20 493	20 520	20 548	20 575	20 602	20 629	20 656		27	26
161	683	710	737	763	790	817	844	871	898	925	1	2.7	2.6
162	952	978	21 005	21 032	21 059	21 085	21 112	21 139	21 165	21 192	2	5.4	5.2
163	21 219	21 245	272	299	325	352	378	405	431	458	3	8.1	7.8
164	484	511	537	564	590	617	643	669	696	722	4	10.8	10.4
165	748	775	801	827	854	880	906	932	958	985	5	13.5	13.0
166	22 011	22 037	22 063	22 089	22 115	22 141	22 167	22 194	22 220	22 246	6	16.2	15.6
167	272	298	324	350	376	401	427	453	479	505	7	18.9	18.2
168	531	557	583	608	634	660	686	712	737	763	8	21.6	20.8
169	789	814	840	866	891	917	943	968	994	23 019	9	24.3	23.4
170	23 045	23 070	23 096	23 121	23 147	23 172	23 198	23 223	23 249	23 274		25	24
171	300	325	350	376	401	426	452	477	502	528	1	2.5	2.4
172	553	578	603	629	654	679	704	729	754	779	2	5.0	4.8
173	805	830	855	880	905	930	955	980	24 005	24 030	3	7.5	7.2
174	24 055	24 080	24 105	24 130	24 155	24 180	24 204	24 229	254	279	4	10.0	9.6
175	304	329	353	378	403	428	452	477	502	527	5	12.5	12.0
176	551	576	601	625	650	674	699	724	748	773	6	15.0	14.4
177	797	822	846	871	895	920	944	969	993	25 018	7	17.5	16.8
178	25 042	25 066	25 091	25 115	25 139	25 164	25 188	25 212	25 237	261	8	20.0	19.2
179	285	310	334	358	382	406	431	455	479	503	9	22.5	21.6
180	25 527	25 551	25 575	25 600	25 624	25 648	25 672	25 696	25 720	25 744		23	22
181	768	792	816	840	864	888	912	935	959	983	1	2.3	2.2
182	26 007	26 031	26 055	26 079	26 102	26 126	26 150	26 174	26 198	26 221	2	4.6	4.4
183	245	269	293	316	340	364	387	411	435	458	3	6.9	6.6
184	482	505	529	553	576	600	623	647	670	694	4	9.2	8.8
185	717	741	764	788	811	834	858	881	905	928	5	11.5	11.0
186	951	975	998	27 021	27 045	27 068	27 091	27 114	27 138	27 161	6	13.8	13.2
187	27 184	27 207	27 231	254	277	300	323	346	370	393	7	16.1	15.4
188	416	439	462	485	508	531	554	577	600	623	8	18.4	17.6
189	646	669	692	715	738	761	784	807	830	852	9	20.7	19.8
190	27 875	27 898	27 921	27 944	27 967	27 989	28 012	28 035	28 058	28 081		21	
191	28 103	28 126	28 149	28 171	28 194	28 217	240	262	285	307	1	2.1	
192	330	353	375	398	421	443	466	488	511	533	2	4.2	
193	556	578	601	623	646	668	691	713	735	758	3	6.3	
194	780	803	825	847	870	892	914	937	959	981	4	8.4	
195	29 003	29 026	29 048	29 070	29 092	29 115	29 137	29 159	29 181	29 203	5	10.5	
196	226	248	270	292	314	336	358	380	403	425	6	12.6	
197	447	469	491	513	535	557	579	601	623	645	7	14.7	
198	667	688	710	732	754	776	798	820	842	863	8	16.8	
199	885	907	929	951	973	994	30 016	30 038	30 060	30 081	9	18.9	
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298			
N	0	1	2	3	4	5	6	7	8	9			

Numbers 200–250 Logs 30103–39950

N	0	1	2	3	4	5	6	7	8	9
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	31 006	31 027	31 048	31 069	31 091	31 112	31 133	31 154
205	31 175	31 197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	32 035	32 056	32 077	32 098	32 118	32 139	32 160	32 181	32 201
210	32 222	32 243	32 263	32 284	32 305	32 325	32 346	32 366	32 387	32 408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	33 001	33 021
214	33 041	33 062	33 082	33 102	33 122	33 143	33 163	33 183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	34 005	34 025
219	34 044	34 064	34 084	34 104	34 124	34 143	34 163	34 183	203	223
220	34 242	34 262	34 282	34 301	34 321	34 341	34 361	34 380	34 400	34 420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	35 005
224	35 025	35 044	35 064	35 083	35 102	35 122	35 141	35 160	35 180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	36 003	36 021	36 040	36 059	36 078	36 097	36 116	36 135	36 154
230	36 173	36 192	36 211	36 229	36 248	36 267	36 286	36 305	36 324	36 342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	37 014	37 033	37 051	37 070	37 088
235	37 107	37 125	37 144	37 162	37 181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	38 003
240	38 021	38 039	38 057	38 075	38 093	38 112	38 130	38 148	38 166	38 184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	39 005	39 023	39 041	39 058	39 076
246	39 094	39 111	39 129	39 146	39 164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950
N	0	1	2	3	4	5	6	7	8	9

P.P.

	22	21	20	19	18	17
1	2.2	2.1	2.0	1.9	1.8	1.7
2	4.4	4.2	4.0	3.8	3.6	3.4
3	6.6	6.3	6.0	5.7	5.4	5.1
4	8.8	8.4	8.0	7.6	7.2	6.8
5	11.0	10.5	10.0	9.5	9.0	8.5
6	13.2	12.6	12.0	11.4	10.8	10.2
7	15.4	14.7	14.0	13.3	12.6	11.9
8	17.6	16.8	16.0	15.2	14.4	13.6
9	19.8	18.9	18.0	17.1	16.2	15.3

Numbers 250–300 Logs 39794–47842

N	0	1	2	3	4	5	6	7	8	9
250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950
251	967	985̄	40 002	40 019	40 037	40 054	40 071	40 088	40 106	40 123
252	40 140	40 157	175̄	192	209	226	243	261	278	295̄
253	312	329	346	364	381	398	415̄	432	449	466
254	483	500	518	535̄	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875̄	892	909	926	943	960	976
257	993	41 010	41 027	41 044	41 061	41 078	41 095̄	41 111	41 128	41 145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	41 497	41 514	41 531	41 547	41 564	41 581	41 597	41 614	41 631	41 647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	42 012	42 029	42 045	42 062	42 078	42 095̄	42 111	42 127	42 144
264	42 160	177	193	210	226	243	259	275	292	308
265	325̄	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635̄
267	651	667	684	700	716	732	749	765̄	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	43 008	43 024	43 040	43 056	43 072	43 088	43 104	43 120
270	43 136	43 152	43 169	43 185	43 201	43 217	43 233	43 249	43 265̄	43 281
271	297	313	329	345̄	361	377	393	409	425̄	441
272	457	473	489	505̄	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965̄	981	996	44 012	44 028	44 044	44 059	44 075
276	44 091	44 107	44 122	44 138	44 154	170	185	201	217	232
277	248	264	279	295̄	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545̄
279	560	576	592	607	623	638	654	669	685̄	700
280	44 716	44 731	44 747	44 762	44 778	44 793	44 809	44 824	44 840	44 855
281	871	886	902	917	932	948	963	979	994	45 010
282	45 025̄	45 040	45 056	45 071	45 086	45 102	45 117	45 133	45 148	163
283	179	194	209	225̄	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515̄	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	46 000	46 015̄	46 030	46 045̄	46 060	46 075̄
289	46 090	46 105̄	46 120	46 135̄	150	165̄	180	195̄	210	225̄
290	46 240	46 255̄	46 270	46 285̄	46 300	46 315̄	46 330	46 345̄	46 359	46 374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835̄	850	864	879	894	909	923	938	953	967
295	982	997	47 012	47 026	47 041	47 056	47 070	47 085	47 100	47 114
296	47 129	47 144	159	173	188	202	217	232	246	261
297	276	290	305̄	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	47 712	47 727	47 741	47 756	47 770	47 784	47 799	47 813	47 828	47 842
N	0	1	2	3	4	5	6	7	8	9

P.P.

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

	16
1	1.6
2	3.2
3	4.8
4	6.4
5	8.0
6	9.6
7	11.2
8	12.8
9	14.4

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

Numbers 300–350 Logs 47712–54518

N	0	1	2	3	4	5	6	7	8	9
300	47 712	47 727	47 741	47 756	47 770	47 784	47 799	47 813	47 828	47 842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	48 015	48 029	48 044	48 058	48 073	48 087	48 101	48 116	48 130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	49 010	49 024	49 038	49 052	49 066	49 080	49 094	49 108	49 122
310	49 136	49 150	49 164	49 178	49 192	49 206	49 220	49 234	49 248	49 262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	50 010	50 024	50 037	50 051	50 065	50 079	50 092
317	50 106	50 120	50 133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	50 515	50 529	50 542	50 556	50 569	50 583	50 596	50 610	50 623	50 637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	51 001	51 014	51 028	51 041
324	51 055	51 068	51 081	51 095	51 108	51 121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	51 851	51 865	51 878	51 891	51 904	51 917	51 930	51 943	51 957	51 970
331	983	996	52 009	52 022	52 035	52 048	52 061	52 075	52 088	52 101
332	52 114	52 127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	53 007
339	53 020	53 033	53 046	53 058	53 071	53 084	53 097	53 110	53 122	135
340	53 148	53 161	53 173	53 186	53 199	53 212	53 224	53 237	53 250	53 263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	54 008	54 020
347	54 033	54 045	54 058	54 070	54 083	54 095	54 108	54 120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518
N	0	1	2	3	4	5	6	7	8	9

P.P.

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Numbers 350–400 Logs 54407–60304

N	0	1	2	3	4	5	6	7	8	9	P.P.
350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518	13
351	531	543	555	568	580	593	605	617	630	642	1 1.3
352	654	667	679	691	704	716	728	741	753	765	2 2.6
353	777	790	802	814	827	839	851	864	876	888	3 3.9
354	900	913	925	937	949	962	974	986	998	55 011	4 5.2
355	55 023	55 035	55 047	55 060	55 072	55 084	55 096	55 108	55 121	133	5 6.5
356	145	157	169	182	194	206	218	230	242	255	6 7.8
357	267	279	291	303	315	328	340	352	364	376	7 9.1
358	388	400	413	425	437	449	461	473	485	497	8 10.4
359	509	522	534	546	558	570	582	594	606	618	9 11.7
360	55 630	55 642	55 654	55 666	55 678	55 691	55 703	55 715	55 727	55 739	12
361	751	763	775	787	799	811	823	835	847	859	1 1.2
362	871	883	895	907	919	931	943	955	967	979	2 2.4
363	991	56 003	56 015	56 027	56 038	56 050	56 062	56 074	56 086	56 098	3 3.6
364	56 110	122	134	146	158	170	182	194	205	217	4 4.8
365	229	241	253	265	277	289	301	312	324	336	5 6.0
366	348	360	372	384	396	407	419	431	443	455	6 7.2
367	467	478	490	502	514	526	538	549	561	573	7 8.4
368	585	597	608	620	632	644	656	667	679	691	8 9.6
369	703	714	726	738	750	761	773	785	797	808	9 10.8
370	56 820	56 832	56 844	56 855	56 867	56 879	56 891	56 902	56 914	56 926	
371	937	949	961	972	984	996	57 008	57 019	57 031	57 043	
372	57 054	57 066	57 078	57 089	57 101	57 113	124	136	148	159	
373	171	183	194	206	217	229	241	252	264	276	
374	287	299	310	322	334	345	357	368	380	392	
375	403	415	426	438	449	461	473	484	496	507	
376	519	530	542	553	565	576	588	600	611	623	
377	634	646	657	669	680	692	703	715	726	738	
378	749	761	772	784	795	807	818	830	841	852	
379	864	875	887	898	910	921	933	944	955	967	
380	57 978	57 990	58 001	58 013	58 024	58 035	58 047	58 058	58 070	58 081	11
381	58 092	58 104	115	127	138	149	161	172	184	195	1 1.1
382	206	218	229	240	252	263	274	286	297	309	2 2.2
383	320	331	343	354	365	377	388	399	410	422	3 3.3
384	433	444	456	467	478	490	501	512	524	535	4 4.4
385	546	557	569	580	591	602	614	625	636	647	5 5.5
386	659	670	681	692	704	715	726	737	749	760	6 6.6
387	771	782	794	805	816	827	838	850	861	872	7 7.7
388	883	894	906	917	928	939	950	961	973	984	8 8.8
389	995	59 006	59 017	59 028	59 040	59 051	59 062	59 073	59 084	59 095	9 9.9
390	59 106	59 118	59 129	59 140	59 151	59 162	59 173	59 184	59 195	59 207	10
391	218	229	240	251	262	273	284	295	306	318	1 1.0
392	329	340	351	362	373	384	395	406	417	428	2 2.0
393	439	450	461	472	483	494	506	517	528	539	3 3.0
394	550	561	572	583	594	605	616	627	638	649	4 4.0
395	660	671	682	693	704	715	726	737	748	759	5 5.0
396	770	780	791	802	813	824	835	846	857	868	6 6.0
397	879	890	901	912	923	934	945	956	966	977	7 7.0
398	988	999	60 010	60 021	60 032	60 043	60 054	60 065	60 076	60 086	8 8.0
399	60 097	60 108	119	130	141	152	163	173	184	195	9 9.0
400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304	
N	0	1	2	3	4	5	6	7	8	9	

Numbers 400–450 Logs 60206–65408

N	0	1	2	3	4	5	6	7	8	9	P.P.
400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304	**11**
401	314	325	336	347	358	369	379	390	401	412	1 1.1
402	423	433	444	455	466	477	487	498	509	520	2 2.2
403	531	541	552	563	574	584	595	606	617	627	3 3.3
404	638	649	660	670	681	692	703	713	724	735	4 4.4
405	746	756	767	778	788	799	810	821	831	842	5 5.5
406	853	863	874	885	895	906	917	927	938	949	6 6.6
407	959	970	981	991	61 002	61 013	61 023	61 034	61 045	61 055	7 7.7
408	61 066	61 077	61 087	61 098	109	119	130	140	151	162	8 8.8
409	172	183	194	204	215	225	236	247	257	268	9 9.9
410	61 278	61 289	61 300	61 310	61 321	61 331	61 342	61 352	61 363	61 374	
411	384	395	405	416	426	437	448	458	469	479	
412	490	500	511	521	532	542	553	563	574	584	
413	595	606	616	627	637	648	658	669	679	690	
414	700	711	721	731	742	752	763	773	784	794	
415	805	815	826	836	847	857	868	878	888	899	
416	909	920	930	941	951	962	972	982	993	62 003	
417	62 014	62 024	62 034	62 045	62 055	62 066	62 076	62 086	62 097	107	
418	118	128	138	149	159	170	180	190	201	211	
419	221	232	242	252	263	273	284	294	304	315	
420	62 325	62 335	62 346	62 356	62 366	62 377	62 387	62 397	62 408	62 418	**10**
421	428	439	449	459	469	480	490	500	511	521	1 1.0
422	531	542	552	562	572	583	593	603	613	624	2 2.0
423	634	644	655	665	675	685	696	706	716	726	3 3.0
424	737	747	757	767	778	788	798	808	818	829	4 4.0
425	839	849	859	870	880	890	900	910	921	931	5 5.0
426	941	951	961	972	982	992	63 002	63 012	63 022	63 033	6 6.0
427	63 043	63 053	63 063	63 073	63 083	63 094	104	114	124	134	7 7.0
428	144	155	165	175	185	195	205	215	225	236	8 8.0
429	246	256	266	276	286	296	306	317	327	337	9 9.0
430	63 347	63 357	63 367	63 377	63 387	63 397	63 407	63 417	63 428	63 438	
431	448	458	468	478	488	498	508	518	528	538	
432	548	558	568	579	589	599	609	619	629	639	
433	649	659	669	679	689	699	709	719	729	739	
434	749	759	769	779	789	799	809	819	829	839	
435	849	859	869	879	889	899	909	919	929	939	
436	949	959	969	979	988	998	64 008	64 018	64 028	64 038	
437	64 048	64 058	64 068	64 078	64 088	64 098	108	118	128	137	
438	147	157	167	177	187	197	207	217	227	237	
439	246	256	266	276	286	296	306	316	326	335	
440	64 345	64 355	64 365	64 375	64 385	64 395	64 404	64 414	64 424	64 434	**9**
441	444	454	464	473	483	493	503	513	523	532	1 .9
442	542	552	562	572	582	591	601	611	621	631	2 1.8
443	640	650	660	670	680	689	699	709	719	729	3 2.7
444	738	748	758	768	777	787	797	807	816	826	4 3.6
445	836	846	856	865	875	885	895	904	914	924	5 4.5
446	933	943	953	963	972	982	992	65 002	65 011	65 021	6 5.4
447	65 031	65 040	65 050	65 060	65 070	65 079	65 089	099	108	118	7 6.3
448	128	137	147	157	167	176	186	196	205	215	8 7.2
449	225	234	244	254	263	273	283	292	302	312	9 8.1
450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408	
N	0	1	2	3	4	5	6	7	8	9	

Numbers 450–500 Logs 65321–69975

N	0	1	2	3	4	5	6	7	8	9	P.P.	
450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408	10	
451	418	427	437	447	456	466	475	485	495	504	1	1.0
452	514	523	533	543	552	562	571	581	591	600	2	2.0
453	610	619	629	639	648	658	667	677	686	696	3	3.0
454	706	715	725	734	744	753	763	772	782	792	4	4.0
455	801	811	820	830	839	849	858	868	877	887	5	5.0
456	896	906	916	925	935	944	954	963	973	982	6	6.0
457	992	66 001	66 011	66 020	66 030	66 039	66 049	66 058	66 068	66 077	7	7.0
458	66 087	096	106	115	124	134	143	153	162	172	8	8.0
459	181	191	200	210	219	229	238	247	257	266	9	9.0
460	66 276	66 285	66 295	66 304	66 314	66 323	66 332	66 342	66 351	66 361		
461	370	380	389	398	408	417	427	436	445	455		
462	464	474	483	492	502	511	521	530	539	549		
463	558	567	577	586	596	605	614	624	633	642		
464	652	661	671	680	689	699	708	717	727	736		
465	745	755	764	773	783	792	801	811	820	829		
466	839	848	857	867	876	885	894	904	913	922		
467	932	941	950	960	969	978	987	997	67 006	67 015		
468	67 025	67 034	67 043	67 052	67 062	67 071	67 080	67 089	099	108		
469	117	127	136	145	154	164	173	182	191	201		
470	67 210	67 219	67 228	67 237	67 247	67 256	67 265	67 274	67 284	67 293	9	
471	302	311	321	330	339	348	357	367	376	385	1	0.9
472	394	403	413	422	431	440	449	459	468	477	2	1.8
473	486	495	504	514	523	532	541	550	560	569	3	2.7
474	578	587	596	605	614	624	633	642	651	660	4	3.6
475	669	679	688	697	706	715	724	733	742	752	5	4.5
476	761	770	779	788	797	806	815	825	834	843	6	5.4
477	852	861	870	879	888	897	906	916	925	934	7	6.3
478	943	952	961	970	979	988	997	68 006	68 015	68 024	8	7.2
479	68 034	68 043	68 052	68 061	68 070	68 079	68 088	097	106	115	9	8.1
480	68 124	68 133	68 142	68 151	68 160	68 169	68 178	68 187	68 196	68 205		
481	215	224	233	242	251	260	269	278	287	296		
482	305	314	323	332	341	350	359	368	377	386		
483	395	404	413	422	431	440	449	458	467	476		
484	485	494	502	511	520	529	538	547	556	565		
485	574	583	592	601	610	619	628	637	646	655		
486	664	673	681	690	699	708	717	726	735	744		
487	753	762	771	780	789	797	806	815	824	833		
488	842	851	860	869	878	886	895	904	913	922		
489	931	940	949	958	966	975	984	993	69 002	69 011		
490	69 020	69 028	69 037	69 046	69 055	69 064	69 073	69 082	69 090	69 099	8	
491	108	117	126	135	144	152	161	170	179	188	1	0.8
492	197	205	214	223	232	241	249	258	267	276	2	1.6
493	285	294	302	311	320	329	338	346	355	364	3	2.4
494	373	381	390	399	408	417	425	434	443	452	4	3.2
495	461	469	478	487	496	504	513	522	531	539	5	4.0
496	548	557	566	574	583	592	601	609	618	627	6	4.8
497	636	644	653	662	671	679	688	697	705	714	7	5.6
498	723	732	740	749	758	767	775	784	793	801	8	6.4
499	810	819	827	836	845	854	862	871	880	888	9	7.2
500	69 897	69 906	69 914	69 923	69 932	69 940	69 949	69 958	69 966	69 975		
N	0	1	2	3	4	5	6	7	8	9		

Numbers 500–550 Logs 69897–74107

N	0	1	2	3	4	5	6	7	8	9	P.P.	
500	69 897	69 906	69 914	69 923	69 932	69 940	69 949	69 958	69 966	69 975	9	
501	984	992	70 001	70 010	70 018	70 027	70 036	70 044	70 053	70 062	1	0.9
502	70 070	70 079	088	096	105	114	122	131	140	148	2	1.8
503	157	165	174	183	191	200	209	217	226	234	3	2.7
504	243	252	260	269	278	286	295	303	312	321	4	3.6
505	329	338	346	355	364	372	381	389	398	406	5	4.5
506	415	424	432	441	449	458	467	475	484	492	6	5.4
507	501	509	518	526	535	544	552	561	569	578	7	6.3
508	586	595	603	612	621	629	638	646	655	663	8	7.2
509	672	680	689	697	706	714	723	731	740	749	9	8.1
510	70 757	70 766	70 774	70 783	70 791	70 800	70 808	70 817	70 825	70 834		
511	842	851	859	868	876	885	893	902	910	919		
512	927	935	944	952	961	969	978	986	995	71 003		
513	71 012	71 020	71 029	71 037	71 046	71 054	71 063	71 071	71 079	088		
514	096	105	113	122	130	139	147	155	164	172		
515	181	189	198	206	214	223	231	240	248	257		
516	265	273	282	290	299	307	315	324	332	341		
517	349	357	366	374	383	391	399	408	416	425		
518	433	441	450	458	466	475	483	492	500	508		
519	517	525	533	542	550	559	567	575	584	592		
520	71 600	71 609	71 617	71 625	71 634	71 642	71 650	71 659	71 667	71 675	8	
521	684	692	700	709	717	725	734	742	750	759	1	0.8
522	767	775	784	792	800	809	817	825	834	842	2	1.6
523	850	858	867	875	883	892	900	908	917	925	3	2.4
524	933	941	950	958	966	975	983	991	999	72 008	4	3.2
525	72 016	72 024	72 032	72 041	72 049	72 057	72 066	72 074	72 082	090	5	4.0
526	099	107	115	123	132	140	148	156	165	173	6	4.8
527	181	189	198	206	214	222	230	239	247	255	7	5.6
528	263	272	280	288	296	304	313	321	329	337	8	6.4
529	346	354	362	370	378	387	395	403	411	419	9	7.2
530	72 428	72 436	72 444	72 452	72 460	72 469	72 477	72 485	72 493	72 501		
531	509	518	526	534	542	550	558	567	575	583		
532	591	599	607	616	624	632	640	648	656	665		
533	673	681	689	697	705	713	722	730	738	746		
534	754	762	770	779	787	795	803	811	819	827		
535	835	843	852	860	868	876	884	892	900	908		
536	916	925	933	941	949	957	965	973	981	989		
537	997	73 006	73 014	73 022	73 030	73 038	73 046	73 054	73 062	73 070		
538	73 078	086	094	102	111	119	127	135	143	151		
539	159	167	175	183	191	199	207	215	223	231		
540	73 239	73 247	73 255	73 263	73 272	73 280	73 288	73 296	73 304	73 312	7	
541	320	328	336	344	352	360	368	376	384	392	1	0.7
542	400	408	416	424	432	440	448	456	464	472	2	1.4
543	480	488	496	504	512	520	528	536	544	552	3	2.1
544	560	568	576	584	592	600	608	616	624	632	4	2.8
545	640	648	656	664	672	679	687	695	703	711	5	3.5
546	719	727	735	743	751	759	767	775	783	791	6	4.2
547	799	807	815	823	830	838	846	854	862	870	7	4.9
548	878	886	894	902	910	918	926	933	941	949	8	5.6
549	957	965	973	981	989	997	74 005	74 013	74 020	74 028	9	6.3
550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107		
N	0	1	2	3	4	5	6	7	8	9		

Numbers 550–600   Logs 74036–77880

N	0	1	2	3	4	5	6	7	8	9	P.P.
550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107	8
551	115	123	131	139	147	155	162	170	178	186	1 0.8
552	194	202	210	218	225	233	241	249	257	265	2 1.6
553	273	280	288	296	304	312	320	327	335	343	3 2.4
554	351	359	367	374	382	390	398	406	414	421	4 3.2
555	429	437	445	453	461	468	476	484	492	500	5 4.0
556	507	515	523	531	539	547	554	562	570	578	6 4.8
557	586	593	601	609	617	624	632	640	648	656	7 5.6
558	663	671	679	687	695	702	710	718	726	733	8 6.4
559	741	749	757	764	772	780	788	796	803	811	9 7.2
560	74 819	74 827	74 834	74 842	74 850	74 858	74 865	74 873	74 881	74 889	
561	896	904	912	920	927	935	943	950	958	966	
562	974	981	989	997	75 005	75 012	75 020	75 028	75 035	75 043	
563	75 051	75 059	75 066	75 074	082	089	097	105	113	120	
564	128	136	143	151	159	166	174	182	189	197	
565	205	213	220	228	236	243	251	259	266	274	
566	282	289	297	305	312	320	328	335	343	351	
567	358	366	374	381	389	397	404	412	420	427	
568	435	442	450	458	465	473	481	488	496	504	
569	511	519	526	534	542	549	557	565	572	580	
570	75 587	75 595	75 603	75 610	75 618	75 626	75 633	75 641	75 648	75 656	7
571	664	671	679	686	694	702	709	717	724	732	1 0.7
572	740	747	755	762	770	778	785	793	800	808	2 1.4
573	815	823	831	838	846	853	861	868	876	884	3 2.1
574	891	899	906	914	921	929	937	944	952	959	4 2.8
575	967	974	982	989	997	76005	76 012	76 020	76 027	76 035	5 3.5
576	76 042	76 050	76 057	76 065	76 072	080	087	095	103	110	6 4.2
577	118	125	133	140	148	155	163	170	178	185	7 4.9
578	193	200	208	215	223	230	238	245	253	260	8 5.6
579	268	275	283	290	298	305	313	320	328	335	9 6.3
580	76 343	76 350	76 358	76 365	76 373	76 380	76 388	76 395	76 403	76 410	
581	418	425	433	440	448	455	462	470	477	485	
582	492	500	507	515	522	530	537	545	552	559	
583	567	574	582	589	597	604	612	619	626	634	
584	641	649	656	664	671	678	686	693	701	708	
585	716	723	730	738	745	753	760	768	775	782	
586	790	797	805	812	819	827	834	842	849	856	
587	864	871	879	886	893	901	908	916	923	930	
588	938	945	953	960	967	975	982	989	997	77 004	
589	77 012	77 019	77 026	77 034	77 041	77 048	77 056	77 063	77 070	078	
590	77 085	77 093	77 100	77 107	77 115	77 122	77 129	77 137	77 144	77 151	
591	159	166	173	181	188	195	203	210	217	225	
592	232	240	247	254	262	269	276	283	291	298	
593	305	313	320	327	335	342	349	357	364	371	
594	379	386	393	401	408	415	422	430	437	444	
595	452	459	466	474	481	488	495	503	510	517	
596	525	532	539	546	554	561	568	576	583	590	
597	597	605	612	619	627	634	641	648	656	663	
598	670	677	685	692	699	706	714	721	728	735	
599	743	750	757	764	772	779	786	793	801	808	
600	77 815	77 822	77 830	77 837	77 844	77 851	77 859	77 866	77 873	77 880	
N	0	1	2	3	4	5	6	7	8	9	

Numbers 600–650 Logs 77815–81351

N	0	1	2	3	4	5	6	7	8	9	P.P.
600	77 815	77 822	77 830	77 837	77 844	77 851	77 859	77 866	77 873	77 880	8
601	887	895	902	909	916	924	931	938	945	952	1 0.8
602	960	967	974	981	988	996	78 003	78 010	78 017	78 025	2 1.6
603	78 032	78 039	78 046	78 053	78 061	78 068	075	082	089	097	3 2.4
604	104	111	118	125	132	140	147	154	161	168	4 3.2
605	176	183	190	197	204	211	219	226	233	240	5 4.0
606	247	254	262	269	276	283	290	297	305	312	6 4.8
607	319	326	333	340	347	355	362	369	376	383	7 5.6
608	390	398	405	412	419	426	433	440	447	455	8 6.4
609	462	469	476	483	490	497	504	512	519	526	9 7.2
610	78 533	78 540	78 547	78 554	78 561	78 569	78 576	78 583	78 590	78 597	
611	604	611	618	625	633	640	647	654	661	668	
612	675	682	689	696	704	711	718	725	732	739	
613	746	753	760	767	774	781	789	796	803	810	
614	817	824	831	838	845	852	859	866	873	880	
615	888	895	902	909	916	923	930	937	944	951	
616	958	965	972	979	986	993	79 000	79 007	79 014	79 021	
617	79 029	79 036	79 043	79 050	79 057	79 064	071	078	085	092	
618	099	106	113	120	127	134	141	148	155	162	
619	169	176	183	190	197	204	211	218	225	232	
620	79 239	79 246	79 253	79 260	79 267	79 274	79 281	79 288	79 295	79 302	7
621	309	316	323	330	337	344	351	358	365	372	1 0.7
622	379	386	393	400	407	414	421	428	435	442	2 1.4
623	449	456	463	470	477	484	491	498	505	511	3 2.1
624	518	525	532	539	546	553	560	567	574	581	4 2.8
625	588	595	602	609	616	623	630	637	644	650	5 3.5
626	657	664	671	678	685	692	699	706	713	720	6 4.2
627	727	734	741	748	754	761	768	775	782	789	7 4.9
628	796	803	810	817	824	831	837	844	851	858	8 5.6
629	865	872	879	886	893	900	906	913	920	927	9 6.3
630	79 934	79 941	79 948	79 955	79 962	79 969	79 975	79 982	79 989	79 996	
631	80 003	80 010	80 017	80 024	80 030	80 037	80 044	80 051	80 058	80 065	
632	072	079	085	092	099	106	113	120	127	134	
633	140	147	154	161	168	175	182	188	195	202	
634	209	216	223	229	236	243	250	257	264	271	
635	277	284	291	298	305	312	318	325	332	339	
636	346	353	359	366	373	380	387	393	400	407	
637	414	421	428	434	441	448	455	462	468	475	
638	482	489	496	502	509	516	523	530	536	543	
639	550	557	564	570	577	584	591	598	604	611	
640	80 618	80 625	80 632	80 638	80 645	80 652	80 659	80 665	80 672	80 679	6
641	686	693	699	706	713	720	726	733	740	747	1 0.6
642	754	760	767	774	781	787	794	801	808	814	2 1.2
643	821	828	835	841	848	855	862	868	875	882	3 1.8
644	889	895	902	909	916	922	929	936	943	949	4 2.4
645	956	963	969	976	983	990	996	81 003	81 010	81 017	5 3.0
646	81 023	81 030	81 037	81 043	81 050	81 057	81 064	070	077	084	6 3.6
647	090	097	104	111	117	124	131	137	144	151	7 4.2
648	158	164	171	178	184	191	198	204	211	218	8 4.8
649	224	231	238	245	251	258	265	271	278	285	9 5.4
650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351	
N	0	1	2	3	4	5	6	7	8	9	

Numbers 650–700 Logs 81291–84566

N	0	1	2	3	4	5	6	7	8	9
650	81 291	81 298	81 305̄	81 311	81 318	81 325̄	81 331	81 338	81 345̄	81 351
651	358	365̄	371	378	385̄	391	398	405̄	411	418
652	425̄	431	438	445̄	451	458	465̄	471	478	485̄
653	491	498	505̄	511	518	525̄	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915̄	921	928	935̄	941	948
660	81 954	81 961	81 968	81 974	81 981	81 987	81 994	82 000	82 007	82 014
661	82 020	82 027	82 033	82 040	82 046	82 053	82 060	066	073	079
662	086	092	099	105	112	119	125	132	138	145̄
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315̄	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465̄	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595̄	601
670	82 607	82 614	82 620	82 627	82 633	82 640	82 646	82 653	82 659	82 666
671	672	679	685	692	698	705̄	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905̄	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995̄	83 001	83 008	83 014	83 020	83 027	83 033	83 040	83 046	83 052
677	83 059	065	072	078	085̄	091	097	104	110	117
678	123	129	136	142	149	155̄	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245̄
680	83 251	83 257	83 264	83 270	83 276	83 283	83 289	83 296	83 302	83 308
681	315̄	321	327	334	340	347	353	359	366	372
682	378	385̄	391	398	404	410	417	423	429	436
683	442	448	455̄	461	467	474	480	487	493	499
684	506	512	518	525̄	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715̄	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835̄	841	847	853	860	866	872	879
690	83 885̄	83 891	83 897	83 904	83 910	83 916	83 923	83 929	83 935	83 942
691	948	954	960	967	973	979	985	992	998	84 004
692	84 011	84 017	84 023	84 029	84 036	84 042	84 048	84 055̄	84 061	067
693	073	080	086	092	098	105̄	111	117	123	130
694	136	142	148	155̄	161	167	173	180	186	192
695	198	205̄	211	217	223	230	236	242	248	255̄
696	261	267	273	280	286	292	298	305̄	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485̄	491	497	504
700	84 510	84 516	84 522	84 528	84 535̄	84 541	84 547	84 553	84 559	84 566
N	0	1	2	3	4	5	6	7	8	9

P.P.

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6	3.6
7	4.2
8	4.8
9	5.4

Numbers 700–750 Logs 84510–87558

N	0	1	2	3	4	5	6	7	8	9
700	84 510	84 516	84 522	84 528	84 535̅	84 541	84 547	84 553	84 559	84 566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665̅	671	677	683	689
703	696	702	708	714	720	726	733	739	745̅	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985̅	991	997
708	85 003	85 009	85 016	85 022	85 028	85 034	85 040	85 046	85 052	85 058
709	065̅	071	077	083	089	095	101	107	114	120
710	85 126	85 132	85 138	85 144	85 150	85 156	85 163	85 169	85 175̅	85 181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285̅	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425̅
715	431	437	443	449	455̅	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625̅	631	637	643	649	655̅	661	667
719	673	679	685̅	691	697	703	709	715	721	727
720	85 733	85 739	85 745	85 751	85 757	85 763	85 769	85 775	85 781	85 788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	86 004	86 010	86 016	86 022	86 028
725	86 034	86 040	86 046	86 052	86 058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255̅	261	267
729	273	279	285̅	291	297	303	308	314	320	326
730	86 332	86 338	86 344	86 350	86 356	86 362	86 368	86 374	86 380	86 386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475̅	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635̅	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735̅	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	86 923	86 929	86 935̅	86 941	86 947	86 953	86 958	86 964	86 970	86 976
741	982	988	994	999	87 005	87 011	87 017	87 023	87 029	87 035
742	87 040	87 046	87 052	87 058	064	070	075	081	087	093
743	099	105̅	111	116	122	128	134	140	146	151
744	157	163	169	175̅	181	186	192	198	204	210
745	216	221	227	233	239	245̅	251	256	262	268
746	274	280	286	291	297	303	309	315̅	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425̅	431	437	442
749	448	454	460	466	471	477	483	489	495̅	500
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558
N	0	1	2	3	4	5	6	7	8	9

P.P.

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8	4.8
9	5.4

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3	1.5
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5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

N	0	1	2	3	4	5	6	7	8	9	P.P.
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558	6
751	564	570	576	581	587	593	599	604	610	616	1 0.6
752	622	628	633	639	645	651	656	662	668	674	2 1.2
753	679	685	691	697	703	708	714	720	726	731	3 1.8
754	737	743	749	754	760	766	772	777	783	789	4 2.4
755	795	800	806	812	818	823	829	835	841	846	5 3.0
756	852	858	864	869	875	881	887	892	898	904	6 3.6
757	910	915	921	927	933	938	944	950	955	961	7 4.2
758	967	973	978	984	990	996	88 001	88 007	88 013	88 018	8 4.8
759	88 024	88 030	88 036	88 041	88 047	88 053	058	064	070	076	9 5.4
760	88 081	88 087	88 093	88 098	88 104	88 110	88 116	88 121	88 127	88 133	
761	138	144	150	156	161	167	173	178	184	190	
762	195	201	207	213	218	224	230	235	241	247	
763	252	258	264	270	275	281	287	292	298	304	
764	309	315	321	326	332	338	343	349	355	360	
765	366	372	377	383	389	395	400	406	412	417	
766	423	429	434	440	446	451	457	463	468	474	
767	480	485	491	497	502	508	513	519	525	530	
768	536	542	547	553	559	564	570	576	581	587	
769	593	598	604	610	615	621	627	632	638	643	
770	88 649	88 655	88 660	88 666	88 672	88 677	88 683	88 689	88 694	88 700	5
771	705	711	717	722	728	734	739	745	750	756	1 0.5
772	762	767	773	779	784	790	795	801	807	812	2 1.0
773	818	824	829	835	840	846	852	857	863	868	3 1.5
774	874	880	885	891	897	902	908	913	919	925	4 2.0
775	930	936	941	947	953	958	964	969	975	981	5 2.5
776	986	992	997	89 003	89 009	89 014	89 020	89 025	89 031	89 037	6 3.0
777	89 042	89 048	89 053	059	064	070	076	081	087	092	7 3.5
778	098	104	109	115	120	126	131	137	143	148	8 4.0
779	154	159	165	170	176	182	187	193	198	204	9 4.5
780	89 209	89 215	89 221	89 226	89 232	89 237	89 243	89 248	89 254	89 260	
781	265	271	276	282	287	293	298	304	310	315	
782	321	326	332	337	343	348	354	360	365	371	
783	376	382	387	393	398	404	409	415	421	426	
784	432	437	443	448	454	459	465	470	476	481	
785	487	492	498	504	509	515	520	526	531	537	
786	542	548	553	559	564	570	575	581	586	592	
787	597	603	609	614	620	625	631	636	642	647	
788	653	658	664	669	675	680	686	691	697	702	
789	708	713	719	724	730	735	741	746	752	757	
790	89 763	89 768	89 774	89 779	89 785	89 790	89 796	89 801	89 807	89 812	
791	818	823	829	834	840	845	851	856	862	867	
792	873	878	883	889	894	900	905	911	916	922	
793	927	933	938	944	949	955	960	966	971	977	
794	982	988	993	998	90 004	90 009	90 015	90 020	90 026	90 031	
795	90 037	90 042	90 048	90 053	059	064	069	075	080	086	
796	091	097	102	108	113	119	124	129	135	140	
797	146	151	157	162	168	173	179	184	189	195	
798	200	206	211	217	222	227	233	238	244	249	
799	255	260	266	271	276	282	287	293	298	304	
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358	
N	0	1	2	3	4	5	6	7	8	9	

Numbers 800–850 Logs 90309–92988

N	0	1	2	3	4	5	6	7	8	9
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358
801	363	369	374	380	38$\bar{5}$	390	396	401	407	412
802	417	423	428	434	439	44$\bar{5}$	450	455	461	466
803	472	477	482	488	493	499	504	509	51$\bar{5}$	520
804	526	531	536	542	547	553	558	563	569	574
805	580	58$\bar{5}$	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	79$\bar{5}$	800	806	811	816	822	827	832	838	843
810	90 849	90 854	90 859	90 86$\bar{5}$	90 870	90 875	90 881	90 886	90 891	90 897
811	902	907	913	918	924	929	934	940	94$\bar{5}$	950
812	956	961	966	972	977	982	988	993	998	91 004
813	91 009	91 014	91 020	91 025	91 030	91 036	91 041	91 046	91 052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	18$\bar{5}$	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	26$\bar{5}$	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	35$\bar{5}$	360	365	371	376
820	91 381	91 387	91 392	91 397	91 403	91 408	91 413	91 418	91 424	91 429
821	434	440	44$\bar{5}$	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	53$\bar{5}$
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	63$\bar{5}$	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	73$\bar{5}$	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	84$\bar{5}$	850
829	855	861	866	871	876	882	887	892	897	903
830	91 908	91 913	91 918	91 924	91 929	91 934	91 939	91 944	91 950	91 95$\bar{5}$
831	960	965	971	976	981	986	991	997	92 002	92 007
832	92 012	92 018	92 023	92 028	92 033	92 038	92 044	92 049	054	059
833	06$\bar{5}$	070	07$\bar{5}$	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	19$\bar{5}$	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	33$\bar{5}$	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	92 428	92 433	92 438	92 443	92 449	92 454	92 459	92 464	92 469	92 474
841	480	48$\bar{5}$	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	64$\bar{5}$	650	65$\bar{5}$	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	84$\bar{5}$	850	85$\bar{5}$	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988
N	0	1	2	3	4	5	6	7	8	9

P.P.

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7	4.2
8	4.8
9	5.4

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3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Numbers 850–900 Logs 92942–95468

N	0	1	2	3	4	5	6	7	8	9	P.P.
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988	6
851	993	998	93 003	93 008	93 013	93 018	93 024	93 029	93 034	93 039	1 0.6
852	93 044	93 049	054	059	064	069	075	080	085	090	2 1.2
853	095	100	105	110	115	120	125	131	136	141	3 1.8
854	146	151	156	161	166	171	176	181	186	192	4 2.4
855	197	202	207	212	217	222	227	232	237	242	5 3.0
856	247	252	258	263	268	273	278	283	288	293	6 3.6
857	298	303	308	313	318	323	328	334	339	344	7 4.2
858	349	354	359	364	369	374	379	384	389	394	8 4.8
859	399	404	409	414	420	425	430	435	440	445	9 5.4
860	93 450	93 455	93 460	93 465	93 470	93 475	93 480	93 485	93 490	93 495	
861	500	505	510	515	520	526	531	536	541	546	
862	551	556	561	566	571	576	581	586	591	596	
863	601	606	611	616	621	626	631	636	641	646	
864	651	656	661	666	671	676	682	687	692	697	
865	702	707	712	717	722	727	732	737	742	747	
866	752	757	762	767	772	777	782	787	792	797	
867	802	807	812	817	822	827	832	837	842	847	
868	852	857	862	867	872	877	882	887	892	897	
869	902	907	912	917	922	927	932	937	942	947	
870	93 952	93 957	93 962	93 967	93 972	93 977	93 982	93 987	93 992	93 997	5
871	94 002	94 007	94 012	94 017	94 022	94 027	94 032	94 037	94 042	94 047	1 0.5
872	052	057	062	067	072	077	082	086	091	096	2 1.0
873	101	106	111	116	121	126	131	136	141	146	3 1.5
874	151	156	161	166	171	176	181	186	191	196	4 2.0
875	201	206	211	216	221	226	231	236	240	245	5 2.5
876	250	255	260	265	270	275	280	285	290	295	6 3.0
877	300	305	310	315	320	325	330	335	340	345	7 3.5
878	349	354	359	364	369	374	379	384	389	394	8 4.0
879	399	404	409	414	419	424	429	433	438	443	9 4.5
880	94 448	94 453	94 458	94 463	94 468	94 473	94 478	94 483	94 488	94 493	
881	498	503	507	512	517	522	527	532	537	542	
882	547	552	557	562	567	571	576	581	586	591	
883	596	601	606	611	616	621	626	630	635	640	
884	645	650	655	660	665	670	675	680	685	689	
885	694	699	704	709	714	719	724	729	734	738	
886	743	748	753	758	763	768	773	778	783	787	
887	792	797	802	807	812	817	822	827	832	836	
888	841	846	851	856	861	866	871	876	880	885	
889	890	895	900	905	910	915	919	924	929	934	
890	94 939	94 944	94 949	94 954	94 959	94 963	94 968	94 973	94 978	94 983	4
891	988	993	998	95 002	95 007	95 012	95 017	95 022	95 027	95 032	1 0.4
892	95 036	95 041	95 046	051	056	061	066	071	075	080	2 0.8
893	085	090	095	100	105	109	114	119	124	129	3 1.2
894	134	139	143	148	153	158	163	168	173	177	4 1.6
895	182	187	192	197	202	207	211	216	221	226	5 2.0
896	231	236	240	245	250	255	260	265	270	274	6 2.4
897	279	284	289	294	299	303	308	313	318	323	7 2.8
898	328	332	337	342	347	352	357	361	366	371	8 3.2
899	376	381	386	390	395	400	405	410	415	419	9 3.6
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468	
N	0	1	2	3	4	5	6	7	8	9	

Numbers 900–950 Logs 95424–97813

N	0	1	2	3	4	5	6	7	8	9
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	54$\bar{5}$	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	66$\bar{5}$	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	78$\bar{5}$	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	89$\bar{5}$	899
910	95 904	95 909	95 914	95 918	95 923	95 928	95 933	95 938	95 942	95 947
911	952	957	961	966	971	976	980	985	990	99$\bar{5}$
912	999	96 004	96 009	96 014	96 019	96 023	96 028	96 033	96 038	96 042
913	96 047	052	057	061	066	071	076	080	085	090
914	09$\bar{5}$	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	18$\bar{5}$
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	27$\bar{5}$	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	36$\bar{5}$	369	374
920	96 379	96 384	96 388	96 393	96 398	96 402	96 407	96 412	96 417	96 421
921	426	431	435	440	44$\bar{5}$	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	52$\bar{5}$	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	60$\bar{5}$	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	68$\bar{5}$	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	75$\bar{5}$	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	82$\bar{5}$	830	834	839	844
930	96 848	96 853	96 858	96 862	96 867	96 872	96 876	96 881	96 886	96 890
931	89$\bar{5}$	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	96$\bar{5}$	970	974	979	984
933	988	993	997	97 002	97 007	97 011	97 016	97 021	97 025	97 030
934	97 03$\bar{5}$	97 039	97 044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	16$\bar{5}$	169
937	174	179	183	188	192	197	202	206	211	216
938	220	22$\bar{5}$	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	97 313	97 317	97 322	97 327	97 331	97 336	97 340	97 345	97 350	97 354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	46$\bar{5}$	470	474	479	483	488	493
944	497	502	506	511	516	520	52$\bar{5}$	529	534	539
945	543	548	552	557	562	566	571	575	580	58$\bar{5}$
946	589	594	598	603	607	612	617	621	626	630
947	63$\bar{5}$	640	644	649	653	658	663	667	672	676
948	681	685	690	69$\bar{5}$	699	704	708	713	717	722
949	727	731	736	740	74$\bar{5}$	749	754	759	763	768
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813
N	0	1	2	3	4	5	6	7	8	9

P.P.

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8	4.0
9	4.5

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7	2.8
8	3.2
9	3.6

Numbers 950–1000 Logs 97772–00039

N	0	1	2	3	4	5	6	7	8	9
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813
951	818	823	827	832	836	841	845	850	855̄	859
952	864	868	873	877	882	886	891	896	900	905̄
953	909	914	918	923	928	932	937	941	946	950
954	955̄	959	964	968	973	978	982	987	991	996
955	98 000	98 005̄	98 009	98 014	98 019	98 023	98 028	98 032	98 037	98 041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105̄	109	114	118	123	127	132
958	137	141	146	150	155̄	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	98 227	98 232	98 236	98 241	98 245	98 250	98 254	98 259	98 263	98 268
961	272	277	281	286	290	295̄	299	304	308	313
962	318	322	327	331	336	340	345̄	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435̄	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525̄	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655̄	659	664	668	673
970	98 677	98 682	98 686	98 691	98 695	98 700	98 704	98 709	98 713	98 717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865̄	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945̄	949	954	958	963	967	972	976	981	985
977	989	994	998	99 003	99 007	99 012	99 016	99 021	99 025	99 029
978	99 034	99 038	99 043	047	052	056	061	065̄	069	074
979	078	083	087	092	096	100	105̄	109	114	118
980	99 123	99 127	99 131	99 136	99 140	99 145̄	99 149	99 154	99 158	99 162
981	167	171	176	180	185̄	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335̄	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445̄	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	99 564	99 568	99 572	99 577	99 581	99 585	99 590	99 594	99 599	99 603
991	607	612	616	621	625̄	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695̄	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765̄	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835̄	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935̄	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	00 004	00 009	00 013	00 017	00 022	00 026	00 030	00 035̄	00 039
N	0	1	2	3	4	5	6	7	8	9

P.P.

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8	3.2
9	3.6

Numbers 1000–1050 Logs 0000000–0215614

N	0	1	2	3	4	5	6	7	8	9
1000	000 0000	0434	0869	1303	1737	2171	260$\bar{5}$	3039	3473	3907
01	4341	477$\bar{5}$	5208	5642	6076	6510	6943	7377	7810	8244
02	8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
03	001 3009	3442	3875	4308	4741	5174	5607	6039	6472	690$\bar{5}$
04	7337	7770	8202	863$\bar{5}$	9067	9499	9932	*0364	*0796	*1228
05	002 1661	2093	252$\bar{5}$	2957	3389	3821	4253	468$\bar{5}$	5116	5548
06	5980	6411	6843	727$\bar{5}$	7706	8138	8569	9001	9432	9863
07	003 029$\bar{5}$	0726	1157	1588	2019	2451	2882	3313	3744	4174
08	4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
09	8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004 3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
11	7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
12	005 1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
13	6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
14	006 0380	0808	1236	1664	2092	2521	2949	3377	380$\bar{5}$	4233
15	4660	5088	5516	5944	6372	6799	7227	765$\bar{5}$	8082	8510
16	8937	936$\bar{5}$	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
17	007 3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
18	7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
19	008 1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020	6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
21	009 0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
22	4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
23	8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
24	010 3000	3424	3848	4272	4696	5120	5544	5967	6391	681$\bar{5}$
25	7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
26	011 1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
27	5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
28	9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
29	012 4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030	8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
31	013 2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
32	6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
33	014 1003	1424	1844	2264	268$\bar{5}$	310$\bar{5}$	3525	3945	4365	4785
34	5205	5625	6045	6465	6885	730$\bar{5}$	772$\bar{5}$	8144	8564	8984
35	9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
36	015 3598	4017	4436	485$\bar{5}$	5274	5693	6112	6531	6950	7369
37	7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
38	016 1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
39	6155	6573	6991	7409	7827	824$\bar{5}$	8663	9080	9498	9916
1040	017 0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
41	4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
42	8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
43	018 2843	3259	3676	4092	4508	492$\bar{5}$	5341	5757	6173	6589
44	700$\bar{5}$	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
45	019 1163	1578	1994	2410	282$\bar{5}$	3240	3656	4071	4486	4902
46	5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
47	9467	9882	*0296	*0711	*1126	*1540	*195$\bar{5}$	*2369	*2784	*3198
48	020 3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
49	775$\bar{5}$	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N	0	1	2	3	4	5	6	7	8	9

Numbers 1050–1100 Logs 0211893–0417479

N	0	1	2	3	4	5	6	7	8	9
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
51	6027	6440	6854	7267	7680	8093	8506	8919	9332	974$\bar{5}$
52	022 0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
53	4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
54	8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
55	023 252$\bar{5}$	2936	3348	3759	4171	4582	4994	5405	5817	6228
56	6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
57	024 0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
58	4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
59	8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025 3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
61	7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
62	026 1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
63	5333	5741	6150	6558	6967	737$\bar{5}$	7783	8192	8600	9008
64	9416	9824	*0233	*0641	*1049	*1457	*186$\bar{5}$	*2273	*2680	*3088
65	027 3496	3904	4312	4719	5127	553$\bar{5}$	5942	6350	6757	716$\bar{5}$
66	7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
67	028 1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
68	5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
69	9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029 3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
71	789$\bar{5}$	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
72	030 1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
73	5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
74	031 0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
75	408$\bar{5}$	4489	4893	5296	5700	6104	6508	6912	7315	7719
76	8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
77	032 2157	2560	2963	3367	3770	4173	4576	4979	5382	578$\bar{5}$
78	6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
79	033 0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080	4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
81	8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
82	034 2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
83	628$\bar{5}$	6686	7087	7487	7888	8289	8690	9091	9491	9892
84	035 0293	0693	1094	149$\bar{5}$	1895	2296	2696	3096	3497	3897
85	4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
86	8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
87	036 2295	269$\bar{5}$	3094	3494	3893	4293	4692	5091	5491	5890
88	6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
89	037 0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090	426$\bar{5}$	4663	5062	5460	5858	6257	665$\bar{5}$	7053	7451	7849
91	8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
92	038 2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
93	6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
94	039 0173	0570	0967	1364	1761	2158	2554	2951	3348	374$\bar{5}$
95	4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
96	8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
97	040 2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
98	6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
99	9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041 3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N	0	1	2	3	4	5	6	7	8	9

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

0°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	. . . . . .		. . . . . .		. . . . . .	10.00 000	60
1	6.46 373	30103	6.46 373	30103	3.53 627	000	59
2	6.76 476	17609	6.76 476	17609	3.23 524	000	58
3	6.94 08$\bar{5}$	12494	6.94 08$\bar{5}$	12494	3.05 915	000	57
4	7.06 579	9691	7.06 579	9691	2.93 421	000	56
5	7.16 270	7918	7.16 270	7918	2.83 730	000	55
6	7.24 188	6694	7.24 188	6694	2.75 812	000	54
7	7.30 882	5800	7.30 882	5800	2.69 118	000	53
8	7.36 682	5115	7.36 682	5115	2.63 318	000	52
9	7.41 797	4576	7.41 797	4576	2.58 203	000	51
10	7.46 373	4139	7.46 373	4139	2.53 627	10.00 000	**50**
11	7.50 512	3779	7.50 512	3779	2.49 488	000	49
12	7.54 291	3476	7.54 291	3476	2.45 709	000	48
13	7.57 767	3218	7.57 767	3219	2.42 233	000	47
14	7.60 985	2997	7.60 986	2996	2.39 014	000	46
15	7.63 982	2802	7.63 982	2803	2.36 018	000	45
16	7.66 784	2633	7.66 78$\bar{5}$	2633	2.33 215	000	44
17	7.69 417	2483	7.69 418	2482	2.30 582	9.99 999	43
18	7.71 900	2348	7.71 900	2348	2.28 100	999	42
19	7.74 248	2227	7.74 248	2228	2.25 752	999	41
20	7.76 475	2119	7.76 476	2119	2.23 524	9.99 999	**40**
21	7.78 594	2021	7.78 595	2020	2.21 40$\bar{5}$	999	39
22	7.80 61$\bar{5}$	1930	7.80 615	1931	2.19 38$\bar{5}$	999	38
23	7.82 545	1848	7.82 546	1848	2.17 454	999	37
24	7.84 393	1773	7.84 394	1773	2.15 606	999	36
25	7.86 166	1704	7.86 167	1704	2.13 833	999	35
26	7.87 870	1639	7.87 871	1639	2.12 129	999	34
27	7.89 509	1579	7.89 510	1579	2.10 490	999	33
28	7.91 088	1524	7.91 089	1524	2.08 911	999	32
29	7.92 612	1472	7.92 613	1473	2.07 387	998	31
30	7.94 084	1424	7.94 086	1424	2.05 914	9.99 998	**30**
31	7.95 508	1379	7.95 510	1379	2.04 490	998	29
32	7.96 887	1336	7.96 889	1336	2.03 111	998	28
33	7.98 223	1297	7.98 225	1297	2.01 77$\bar{5}$	998	27
34	7.99 520	1259	7.99 522	1259	2.00 478	998	26
35	8.00 779	1223	8.00 781	1223	1.99 219	998	25
36	8.02 002	1190	8.02 004	1190	1.97 996	998	24
37	8.03 192	1158	8.03 194	1159	1.96 806	997	23
38	8.04 350	1128	8.04 353	1128	1.95 647	997	22
39	8.05 478	1100	8.05 481	1100	1.94 519	997	21
40	8.06 578	1072	8.06 581	1072	1.93 419	9.99 997	**20**
41	8.07 650	1046	8.07 653	1047	1.92 347	997	19
42	8.08 696	1022	8.08 700	1022	1.91 300	997	18
43	8.09 718	999	8.09 722	998	1.90 278	997	17
44	8.10 717	976	8.10 720	976	1.89 280	996	16
45	8.11 693	954	8.11 696	955	1.88 304	996	15
46	8.12 647	934	8.12 651	934	1.87 349	996	14
47	8.13 581	914	8.13 585	915	1.86 41$\bar{5}$	996	13
48	8.14 495	896	8.14 500	895	1.85 500	996	12
49	8.15 391	877	8.15 395	878	1.84 60$\bar{5}$	996	11
50	8.16 268	860	8.16 273	860	1.83 727	9.99 995	**10**
51	8.17 128	843	8.17 133	843	1.82 867	995	9
52	8.17 971	827	8.17 976	828	1.82 024	995	8
53	8.18 798	812	8.18 804	812	1.81 196	99$\bar{5}$	7
54	8.19 610	797	8.19 616	797	1.80 384	99$\bar{5}$	6
55	8.20 407	782	8.20 413	782	1.79 587	994	5
56	8.21 189	769	8.21 195	769	1.78 80$\bar{5}$	994	4
57	8.21 958	755	8.21 964	756	1.78 036	994	3
58	8.22 713	743	8.22 720	742	1.77 280	994	2
59	8.23 456	730	8.23 462	730	1.76 538	994	1
60	8.24 186		8.24 192		1.75 808	9.99 993	**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.			
1′	5800	4576	3779
10″	967	763	630
20″	1933	1525	1260
30″	2900	2288	1890
40″	3867	3051	2519
50″	4833	3813	3149
1′	3218	2802	2483
10″	536	467	414
20″	1073	934	828
30″	1609	1401	1242
40″	2145	1868	1655
50″	2682	2335	2069
1′	2227	2021	1848
10″	371	337	308
20″	742	674	616
30″	1114	1011	924
40″	1485	1347	1232
50″	1856	1684	1540
1′	1704	1579	1472
10″	284	263	245
20″	568	526	491
30″	852	790	736
40″	1136	1053	981
50″	1420	1316	1227
1′	1379	1297	1223
10″	230	216	204
20″	460	432	408
30″	690	649	612
40″	919	865	815
50″	1149	1081	1019
1′	1158	1100	1046
10″	193	183	174
20″	386	367	349
30″	579	550	523
40″	772	733	697
50″	965	917	872
1′	999	954	914
10″	167	159	152
20″	333	318	305
30″	500	477	457
40″	666	636	609
50″	833	795	762
1′	877	843	812
10″	146	141	135
20″	292	281	271
30″	439	422	406
40″	585	562	541
50″	731	703	677
1′	782	755	730
10″	130	126	122
20″	261	252	243
30″	391	378	365
40″	521	503	487
50″	652	629	608

89°

Logarithmic Functions

1°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	8.24 186	717	8.24 192	718	1.75 808	9.99 993	60
1	903	706	910	706	090	993	59
2	8.25 609	695	8.25 616	696	1.74 384	993	58
3	8.26 304	684	8.26 312	684	1.73 688	993	57
4	988	673	996	673	004	992	56
5	8.27 661	663	8.27 669	663	1.72 331	992	55
6	8.28 324	653	8.28 332	654	1.71 668	992	54
7	977	644	986	643	014	992	53
8	8.29 62$\underline{1}$	634	8.29 629	634	1.70 371	992	52
9	8.30 25$\bar{5}$	624	8.30 263	625	1.69 737	991	51
10	8.30 879	616	8.30 888	617	1.69 112	9.99 991	50
11	8.31 495	608	8.31 50$\bar{5}$	607	1.68 495	991	49
12	8.32 103	599	8.32 112	599	1.67 888	990	48
13	702	590	711	591	289	990	47
14	8.33 292	583	8.33 302	584	1.66 698	990	46
15	875	575	886	575	114	990	45
16	8.34 450	568	8.34 461	568	1.65 539	989	44
17	8.35 018	560	8.35 029	561	1.64 971	989	43
18	578	553	590	553	410	989	42
19	8.36 131	547	8.36 143	546	1.63 857	989	41
20	8.36 678	539	8.36 689	540	1.63 311	9.99 988	40
21	8.37 217	533	8.37 229	533	1.62 771	988	39
22	750	526	762	527	238	988	38
23	8.38 276	520	8.38 289	520	1.61 711	987	37
24	796	514	809	514	191	987	36
25	8.39 310	508	8.39 323	509	1.60 677	987	35
26	818	502	832	502	168	986	34
27	8.40 320	496	8.40 334	496	1.59 666	986	33
28	816	491	830	491	170	986	32
29	8.41 307	485	8.41 321	486	1.58 679	985	31
30	8.41 792	480	8.41 807	480	1.58 193	9.99 985	30
31	8.42 272	474	8.42 287	475	1.57 713	98$\bar{5}$	29
32	746	470	762	470	238	984	28
33	8.43 216	464	8.43 232	464	1.56 768	984	27
34	680	459	696	460	304	984	26
35	8.44 139	455	8.44 156	455	1.55 844	983	25
36	594	450	611	450	389	983	24
37	8.45 044	445	8.45 061	446	1.54 939	983	23
38	489	441	507	441	493	982	22
39	930	436	94$\underline{8}$	437	052	982	21
40	8.46 366	433	8.46 38$\bar{5}$	432	1.53 615	9.99 982	20
41	799	427	817	428	18$\underline{3}$	981	19
42	8.47 226	424	8.47 245	424	1.52 75$\bar{5}$	981	18
43	650	419	669	420	331	981	17
44	8.48 06$\underline{9}$	416	8.48 089	416	1.51 91$\underline{1}$	980	16
45	48$\bar{5}$	411	505	412	49$\bar{5}$	980	15
46	896	408	917	408	08$\underline{3}$	979	14
47	8.49 304	404	8.49 325	404	1.50 67$\bar{5}$	979	13
48	708	400	729	401	271	979	12
49	8.50 108	396	8.50 130	397	1.49 870	978	11
50	8.50 504	393	8.50 527	393	1.49 473	9.99 978	10
51	897	390	920	390	080	977	9
52	8.51 287	386	8.51 310	386	1.48 690	977	8
53	673	382	696	383	304	977	7
54	8.52 055	379	8.52 079	380	1.47 921	976	6
55	434	376	459	376	541	976	5
56	810	373	83$\bar{5}$	373	165	97$\underline{5}$	4
57	8.53 183	369	8.53 208	370	1.46 792	97$\bar{5}$	3
58	552	367	57$\underline{8}$	367	422	974	2
59	919	363	94$\bar{5}$	363	055	974	1
60	8.54 282		8.54 308		1.45 692	9.99 974	0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.

1′	717	673	644
10″	120	112	107
20″	239	224	215
30″	359	337	322
40″	478	449	429
50″	598	561	537
1′	616	590	568
10″	103	98	95
20″	205	197	189
30″	308	295	284
40″	411	393	379
50″	513	492	473
1′	547	533	520
10″	91	89	87
20″	182	178	173
30″	274	267	260
40″	365	355	347
50″	456	444	433
1′	508	496	485
10″	85	83	81
20″	169	165	162
30″	254	248	243
40″	339	331	323
50″	423	413	404
1′	474	464	455
10″	79	77	76
20″	158	155	152
30″	237	232	228
40″	316	309	303
50″	395	387	379
1′	445	436	427
10″	74	73	71
20″	148	145	142
30″	223	218	214
40″	297	291	285
50″	371	363	356
1′	419	411	404
10″	70	69	67
20″	140	137	135
30″	210	206	202
40″	279	274	269
50″	349	343	337
1′	396	390	382
10″	66	65	64
20″	132	130	127
30″	198	195	191
40″	264	260	255
50″	330	325	318
1′	376	369	363
10″	63	62	61
20″	125	123	121
30″	188	185	182
40″	251	246	242
50″	313	308	303

88°

2°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	8.54 282	360	8.54 308	361	1.45 692	9.99 974	60
1	642	357	669	358	331	973	59
2	999	355	8.55 027	355	1.44 973	973	58
3	8.55 354	351	382	352	618	972	57
4	705	349	734	349	266	972	56
5	8.56 054	346	8.56 083	346	1.43 917	971	55
6	400	343	429	344	571	971	54
7	743	341	773	341	227	970	53
8	8.57 084	337	8.57 114	338	1.42 886	970	52
9	421	336	452	336	548	969	51
10	8.57 757	332	8.57 788	333	1.42 212	9.99 969	**50**
11	8.58 089	330	8.58 121	330	1.41 879	968	49
12	419	328	451	328	549	968	48
13	747	325	779	326	22$\underline{1}$	967	47
14	8.59 07$\underline{2}$	323	8.59 105	323	1.40 89$\bar{5}$	967	46
15	39$\bar{5}$	320	428	321	572	967	45
16	715	318	749	319	251	966	44
17	8.60 033	316	8.60 068	316	1.39 932	96$\underline{6}$	43
18	349	313	384	314	616	96$\bar{5}$	42
19	662	311	698	311	302	964	41
20	8.60 973	309	8.61 009	310	1.38 991	9.99 964	**40**
21	8.61 282	307	319	307	681	963	39
22	589	305	626	305	374	963	38
23	894	302	931	303	069	962	37
24	8.62 196	301	8.62 234	301	1.37 76$\underline{6}$	962	36
25	497	298	535	299	46$\bar{5}$	961	35
26	79$\bar{5}$	296	834	297	166	961	34
27	8.63 091	294	8.63 131	295	1.36 869	960	33
28	385	293	426	292	574	960	32
29	678	290	718	291	282	959	31
30	8.63 968	288	8.64 009	289	1.35 991	9.99 959	**30**
31	8.64 256	287	298	287	70$\underline{2}$	958	29
32	543	284	585	285	41$\bar{5}$	958	28
33	827	283	870	284	130	957	27
34	8.65 110	281	8.65 154	281	1.34 84$\underline{6}$	956	26
35	391	279	435	280	56$\bar{5}$	956	25
36	670	277	71$\bar{5}$	278	285	95$\underline{5}$	24
37	947	276	993	276	007	95$\bar{5}$	23
38	8.66 223	274	8.66 269	274	1.33 731	954	22
39	497	272	543	273	457	954	21
40	8.66 769	270	8.66 816	271	1.33 184	9.99 953	**20**
41	8.67 039	269	8.67 087	269	1.32 913	952	19
42	308	267	356	268	644	952	18
43	575	266	624	266	376	951	17
44	841	263	890	264	110	951	16
45	8.68 104	263	8.68 154	263	1.31 846	950	15
46	367	260	417	261	583	949	14
47	627	259	678	260	322	949	13
48	886	258	938	258	062	948	12
49	8.69 144	256	8.69 196	257	1.30 804	948	11
50	8.69 400	254	8.69 453	255	1.30 547	9.99 947	**10**
51	654	253	708	254	292	946	9
52	907	252	962	252	038	94$\underline{6}$	8
53	8.70 159	250	8.70 21$\underline{4}$	251	1.29 786	94$\bar{5}$	7
54	409	249	46$\bar{5}$	249	535	944	6
55	658	247	714	248	286	944	5
56	90$\bar{5}$	246	962	246	038	943	4
57	8.71 151	244	8.71 208	245	1.28 792	942	3
58	395	243	453	244	547	942	2
59	638	242	697	243	303	941	1
60	8.71 880		8.71 940		1.28 060	9.99 940	**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.			
1′	360	350	340
1″	6.0	5.8	5.7
2″	12.0	11.7	11.3
3″	18.0	17.5	17.0
4″	24.0	23.3	22.7
5″	30.0	29.2	28.3
6″	36.0	35.0	34.0
7″	42.0	40.8	39.7
8″	48.0	46.7	45.3
9″	54.0	52.5	51.0
1′	330	320	310
1″	5.5	5.3	5.2
2″	11.0	10.7	10.3
3″	16.5	16.0	15.5
4″	22.0	21.3	20.7
5″	27.5	26.7	25.8
6″	33.0	32.0	31.0
7″	38.5	37.3	36.2
8″	44.0	42.7	41.3
9″	49.5	48.0	46.5
1′	300	290	285
1″	5.0	4.8	4.8
2″	10.0	9.7	9.5
3″	15.0	14.5	14.3
4″	20.0	19.3	19.0
5″	25.0	24.2	23.8
6″	30.0	29.0	28.5
7″	35.0	33.8	33.3
8″	40.0	38.7	38.0
9″	45.0	43.5	42.8
1′	280	275	270
1″	4.7	4.6	4.5
2″	9.3	9.2	9.0
3″	14.0	13.8	13.5
4″	18.7	18.3	18.0
5″	23.3	22.9	22.5
6″	28.0	27.5	27.0
7″	32.7	32.1	31.5
8″	37.3	36.7	36.0
9″	42.0	41.3	40.5
1′	265	260	255
1″	4.4	4.3	4.3
2″	8.8	8.7	8.5
3″	13.3	13.0	12.8
4″	17.7	17.3	17.0
5″	22.1	21.7	21.3
6″	26.5	26.0	25.5
7″	30.9	30.3	29.8
8″	35.3	34.7	34.0
9″	39.8	39.0	38.3
1′	250	245	240
1″	4.2	4.1	4.0
2″	8.3	8.2	8.0
3″	12.5	12.3	12.0
4″	16.7	16.3	16.0
5″	20.8	20.4	20.0
6″	25.0	24.5	24.0
7″	29.2	28.6	28.0
8″	33.3	32.7	32.0
9″	37.5	36.8	36.0

87°

3°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	1.27 819	940	59
2	359	238	420	239	580	939	58
3	597	237	659	237	341	938	57
4	834	235	896	236	104	938	56
5	8.73 069	234	8.73 132	234	1.26 868	937	55
6	303	232	366	234	634	936	54
7	535	232	600	232	400	936	53
8	767	230	832	231	168	935	52
9	997	229	8.74 063	229	1.25 937	934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	**50**
11	454	226	521	227	479	933	49
12	680	226	748	226	252	932	48
13	906	224	974	225	026	932	47
14	8.75 130	223	8.75 199	224	1.24 801	931	46
15	353	222	423	222	577	930	45
16	57$\bar{5}$	220	645	222	35$\bar{5}$	929	44
17	795	220	867	220	133	929	43
18	8.76 015	219	8.76 087	219	1.23 913	928	42
19	234	217	306	219	694	927	41
20	8.76 451	216	8.76 525	217	1.23 475	9.99 926	**40**
21	667	216	742	216	258	92$\underline{6}$	39
22	883	214	958	215	042	92$\bar{5}$	38
23	8.77 097	213	8.77 173	214	1.22 827	924	37
24	310	212	387	213	613	923	36
25	522	211	600	211	400	923	35
26	733	210	811	211	189	922	34
27	943	209	8.78 022	210	1.21 978	921	33
28	8.78 152	208	232	209	768	920	32
29	360	208	441	208	559	920	31
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	**30**
31	774	205	855	206	14$\bar{5}$	918	29
32	979	204	8.79 061	205	1.20 939	917	28
33	8.79 183	203	266	204	734	917	27
34	386	202	470	203	530	916	26
35	588	201	673	202	327	915	25
36	789	201	875	201	12$\bar{5}$	914	24
37	990	199	8.80 076	201	1.19 924	913	23
38	8.80 189	199	277	199	723	913	22
39	388	197	476	198	524	912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	**20**
41	782	196	872	196	128	910	19
42	978	195	8.81 068	196	1.18 932	909	18
43	8.81 173	194	264	195	736	909	17
44	367	193	459	194	541	908	16
45	560	192	653	193	347	907	15
46	752	192	846	192	154	906	14
47	944	190	8.82 038	192	1.17 962	905	13
48	8.82 134	190	230	190	770	904	12
49	324	189	420	190	580	904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	**10**
51	701	187	799	188	201	902	9
52	88$\underline{8}$	187	98$\underline{7}$	188	013	901	8
53	8.83 07$\bar{5}$	186	8.83 17$\bar{5}$	186	1.16 825	900	7
54	261	185	361	186	639	899	6
55	446	184	547	185	453	898	5
56	630	183	732	184	268	898	4
57	813	183	916	184	084	897	3
58	996	181	8.84 100	182	1.15 900	89$\underline{6}$	2
59	8.84 177	181	282	182	718	89$\bar{5}$	1
60	8.84 358		8.84 464		1.15 536	9.99 894	**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.			
1′	241	237	233
1″	4.0	4.0	3.9
2″	8.0	7.9	7.8
3″	12.1	11.9	11.7
4″	16.1	15.8	15.5
5″	20.1	19.8	19.4
6″	24.1	23.7	23.3
7″	28.1	27.7	27.2
8″	32.1	31.6	31.1
9″	36.2	35.6	35.0
1′	229	225	221
1″	3.8	3.8	3.7
2″	7.6	7.5	7.4
3″	11.5	11.3	11.1
4″	15.3	15.0	14.7
5″	19.1	18.8	18.4
6″	22.9	22.5	22.1
7″	26.7	26.3	25.8
8″	30.5	30.0	29.5
9″	34.4	33.8	33.2
1′	217	213	209
1″	3.6	3.6	3.5
2″	7.2	7.1	7.0
3″	10.9	10.7	10.5
4″	14.5	14.2	13.9
5″	18.1	17.8	17.4
6″	21.7	21.3	20.9
7″	25.3	24.9	24.4
8″	28.9	28.4	27.9
9″	32 6	32.0	31.4
1′	205	202	199
1″	3.4	3.4	3.3
2″	6.8	6.7	6.6
3″	10.3	10.1	10.0
4″	13.7	13.5	13.3
5″	17.1	16.8	16.6
6″	20.5	20.2	19.9
7″	23.9	23.6	23.2
8″	27.3	26.9	26.5
9″	30.8	30.3	29.9
1′	196	193	190
1″	3.3	3.2	3.2
2″	6.5	6.4	6.3
3″	9.8	9.7	9.5
4″	13.1	12.9	12.7
5″	16.3	16.1	15.8
6″	19.6	19.3	19.0
7″	22.9	22.5	22.2
8″	26.1	25.7	25.3
9″	29.4	29.0	28.5
1′	187	184	181
1″	3.1	3.1	3.0
2″	6.2	6.1	6.0
3″	9.4	9.2	9.1
4″	12.5	12.3	12.1
5″	15.6	15.3	15.1
6″	18.7	18.4	18.1
7″	21.8	21.5	21.1
8″	24.9	24.5	24.1
9″	28.1	27.6	27.2

86°

4°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	**60**
1	539	179	646	180	354	893	59
2	718	179	826	180	174	892	58
3	897	178	8.85 006	179	1.14 994	891	57
4	8.85 075	177	185	178	815	891	56
5	252	177	363	177	637	890	55
6	429	176	540	177	460	889	54
7	605	175	717	176	283	888	53
8	780	175	893	176	107	887	52
9	955	173	8.86 069	174	1.13 931	886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	**50**
11	301	173	417	174	583	884	49
12	474	171	591	172	409	883	48
13	645	171	763	172	237	882	47
14	816	171	935	171	065	881	46
15	987	169	8.87 106	171	1.12 894	880	45
16	8.87 156	169	277	170	723	879	44
17	325	169	447	169	553	879	43
18	494	167	616	169	384	878	42
19	661	168	785	168	215	877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	**40**
21	995	166	8.88 120	167	1.11 880	875	39
22	8.88 161	165	287	166	713	874	38
23	326	164	453	165	547	873	37
24	490	164	618	165	382	872	36
25	654	163	783	165	217	871	35
26	817	163	948	163	052	870	34
27	980	162	8.89 111	163	1.10 889	869	33
28	8.89 142	162	274	163	726	868	32
29	304	160	437	161	563	867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	**30**
31	625	159	760	160	240	865	29
32	784	159	920	160	080	864	28
33	943	159	8.90 080	160	1.09 920	863	27
34	8.90 102	158	240	159	760	862	26
35	260	157	399	158	601	861	25
36	417	157	557	158	443	860	24
37	574	156	715	157	285	859	23
38	730	155	872	157	128	858	22
39	885	155	8.91 029	156	1.08 971	857	21
40	8.91 040	155	8.91 185	155	1.08 815	9.99 856	**20**
41	195	154	340	155	660	855	19
42	349	153	495	155	505	854	18
43	502	153	650	153	350	853	17
44	655	152	803	154	197	852	16
45	807	152	957	153	043	851	15
46	959	151	8.92 110	152	1.07 890	850	14
47	8.92 110	151	262	152	738	848	13
48	261	150	414	151	586	847	12
49	411	150	565	151	435	846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	**10**
51	710	149	866	150	134	844	9
52	859	148	8.93 016	149	1.06 984	843	8
53	8.93 007	147	165	148	835	842	7
54	154	147	313	149	687	841	6
55	301	147	462	147	538	840	5
56	448	146	609	147	391	839	4
57	594	146	756	147	244	838	3
58	740	145	903	146	097	837	2
59	885	145	8.94 049	146	1.05 951	836	1
60	8.94 030		8.94 195		1.05 805	9.99 834	**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.			
1′	181	178	175
1″	3.0	3.0	2.9
2″	6.0	5.9	5.8
3″	9.1	8.9	8.8
4″	12.1	11.9	11.7
5″	15.1	14.8	14.6
6″	18.1	17.8	17.5
7″	21.1	20.8	20.4
8″	24.1	23.7	23.3
9″	27.2	26.7	26.3
1′	173	171	169
1″	2.9	2.9	2.8
2″	5.8	5.7	5.6
3″	8.7	8.6	8.5
4″	11.5	11.4	11.3
5″	14.4	14.3	14.1
6″	17.3	17.1	16.9
7″	20.2	20.0	19.7
8″	23.1	22.8	22.5
9″	26.0	25.7	25.4
1′	167	165	163
1″	2.8	2.8	2.7
2″	5.6	5.5	5.4
3″	8.4	8.3	8.2
4″	11.1	11.0	10.9
5″	13.9	13.8	13.6
6″	16.7	16.5	16.3
7″	19.5	19.3	19.0
8″	22.3	22.0	21.7
9″	25.1	24.8	24.5
1′	161	159	157
1″	2.7	2.7	2.6
2″	5.4	5.3	5.2
3″	8.1	8.0	7.9
4″	10.7	10.6	10.5
5″	13.4	13.3	13.1
6″	16.1	15.9	15.7
7″	18.8	18.6	18.3
8″	21.5	21.2	20.9
9″	24.2	23.9	23.6
1′	155	153	151
1″	2.6	2.6	2.5
2″	5.2	5.1	5.0
3″	7.8	7.7	7.6
4″	10.3	10.2	10.1
5″	12.9	12.8	12.6
6″	15.5	15.3	15.1
7″	18.1	17.9	17.6
8″	20.7	20.4	20.1
9″	23.3	23.0	22.7
1′	149	147	145
1″	2.5	2.5	2.4
2″	5.0	4.9	4.8
3″	7.5	7.4	7.3
4″	9.9	9.8	9.7
5″	12.4	12.3	12.1
6″	14.9	14.7	14.5
7″	17.4	17.2	16.9
8″	19.9	19.6	19.3
9″	22.4	22.1	21.8

85°

5°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	8.94 030	144	8.94 195	145	1.05 805̄	9.99 834	**60**
1	174	143	340	145	660	833	59
2	317	144	485	145	515̄	832	58
3	461	142	630	143	370	831	57
4	603	143	773	144	227	830	56
5	746	141	917	143	083	829	55
6	887	142	8.95 060	142	1.04 940	828	54
7	8.95 029	141	202	142	798	827	53
8	170	140	344	142	656	825	52
9	310	140	486	141	514	824	51
10	8.95 450	139	8.95 627	140	1.04 373	9.99 823	**50**
11	589	139	767	141	233	822	49
12	728	139	908	139	092	821	48
13	867	138	8.96 047	140	1.03 953	820	47
14	8.96 005	138	187	138	813	819	46
15	143	137	325	139	675̄	817	45
16	280	137	464	138	536	816	44
17	417	136	602	137	398	815	43
18	553	136	739	138	261	814	42
19	689	136	877	136	123	813	41
20	8.96 825̄	135	8.97 013	137	1.02 987	9.99 812	**40**
21	960	135	150	135	850	810	39
22	8.97 095̄	134	285	136	715̄	809	38
23	229	134	421	135	579	808	37
24	363	133	556	135	444	807	36
25	496	133	691	134	309	806	35
26	629	133	825̄	134	175	804	34
27	762	132	959	133	041	803	33
28	894	132	8.98 092	133	1.01 908	802	32
29	8.98 026	131	225	133	775̄	801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	**30**
31	288	131	490	132	510	798	29
32	419	130	622	131	378	797	28
33	549	130	753	131	247	796	27
34	679	129	884	131	116	795̄	26
35	808	129	8.99 015̄	130	1.00 985	793	25
36	937	129	145	130	855̄	792	24
37	8.99 066	128	275	130	725̄	791	23
38	194	128	405̄	129	595	790	22
39	322	128	534	128	466	788	21
40	8.99 450	127	8.99 662	129	1.00 338	9.99 787	**20**
41	577	127	791	128	209	786	19
42	704	126	919	127	081	785̄	18
43	830	126	9.00 046	128	0.99 954	783	17
44	956	126	174	127	826	782	16
45	9.00 082	125	301	126	699	781	15
46	207	125	427	126	573	780	14
47	332	124	553	126	447	778	13
48	456	125	679	126	321	777	12
49	581	123	805̄	125	195	776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775̄	**10**
51	828	123	9.01 055̄	124	0.98 945	773	9
52	951	123	179	124	821	772	8
53	9.01 074	122	303	124	697	771	7
54	196	122	427	123	573	769	6
55	318	122	550	123	450	768	5
56	440	121	673	123	327	767	4
57	561	121	796	122	204	765	3
58	682	121	918	122	082	764	2
59	803	120	9.02 040	122	0.97 960	763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.			
1′	145	143	141
1″	2.4	2.4	2.4
2″	4.8	4.8	4.7
3″	7.3	7.2	7.1
4″	9.7	9.5	9.4
5″	12.1	11.9	11.8
6″	14.5	14.3	14.1
7″	16.9	16.7	16.5
8″	19.3	19.1	18.8
9″	21.8	21.5	21.2
1′	139	137	135
1″	2.3	2.3	2.3
2″	4.6	4.6	4.5
3″	7.0	6.9	6.8
4″	9.3	9.1	9.0
5″	11.6	11.4	11.3
6″	13.9	13.7	13.5
7″	16.2	16.0	15.8
8″	18.5	18.3	18.0
9″	20.9	20.6	20.3
1′	133	131	129
1″	2.2	2.2	2.2
2″	4.4	4.4	4.3
3″	6.7	6.6	6.5
4″	8.9	8.7	8.6
5″	11.1	10.9	10.8
6″	13.3	13.1	12.9
7″	15.5	15.3	15.1
8″	17.7	17.5	17.2
9″	20.0	19.7	19.4
1′	128	127	126
1″	2.1	2.1	2.1
2″	4.3	4.2	4.2
3″	6.4	6.4	6.3
4″	8.5	8.5	8.4
5″	10.7	10.6	10.5
6″	12.8	12.7	12.6
7″	14.9	14.8	14.7
8″	17.1	16.9	16.8
9″	19.2	19.1	18.9
1′	125	124	123
1″	2.1	2.1	2.1
2″	4.2	4.1	4.1
3″	6.3	6.2	6.2
4″	8.3	8.3	8.2
5″	10.4	10.3	10.3
6″	12.5	12.4	12.3
7″	14.6	14.5	14.4
8″	16.7	16.5	16.4
9″	18.8	18.6	18.5
1′	122	121	120
1″	2.0	2.0	2.0
2″	4.1	4.0	4.0
3″	6.1	6.1	6.0
4″	8.1	8.1	8.0
5″	10.2	10.1	10.0
6″	12.2	12.1	12.0
7″	14.2	14.1	14.0
8″	16.3	16.1	16.0
9″	18.3	18.2	18.0

84°

Logarithmic Functions

6°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	60
1	9.02 043	120	283	121	717	760	59
2	163	120	404	121	596	759	58
3	283	119	525	120	475	757	57
4	402	118	645	121	355	756	56
5	520	119	766	119	234	755	55
6	639	118	885	120	115	753	54
7	757	117	9.03 005	119	0.96 995	752	53
8	874	118	124	118	876	751	52
9	992	117	242	119	758	749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	50
11	226	116	479	118	521	747	49
12	342	116	597	117	403	745	48
13	458	116	714	118	286	744	47
14	574	116	832	116	168	742	46
15	690	115	948	117	052	741	45
16	805	115	9.04 065	116	0.95 935	740	44
17	920	114	181	116	819	738	43
18	9.04 034	115	297	116	703	737	42
19	149	113	413	115	587	736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	40
21	376	114	643	115	357	733	39
22	490	113	758	115	242	731	38
23	603	112	873	114	127	730	37
24	715	113	987	114	013	728	36
25	828	112	9.05 101	113	0.94 899	727	35
26	940	112	214	114	786	726	34
27	9.05 052	112	328	113	672	724	33
28	164	111	441	112	559	723	32
29	275	111	553	113	447	721	31
30	9.05 386	111	9.05 666	112	0.94 334	9.99 720	30
31	497	110	778	112	222	718	29
32	607	110	890	112	110	717	28
33	717	110	9.06 002	111	0.93 998	716	27
34	827	110	113	111	887	714	26
35	937	109	224	111	776	713	25
36	9.06 046	109	335	110	665	711	24
37	155	109	445	111	555	710	23
38	264	108	556	110	444	708	22
39	372	109	666	109	334	707	21
40	9.06 481	108	9.06 775	110	0.93 225	9.99 705	20
41	589	107	885	109	115	704	19
42	696	108	994	109	006	702	18
43	804	107	9.07 103	108	0.92 897	701	17
44	911	107	211	109	789	699	16
45	9.07 018	106	320	108	680	698	15
46	124	107	428	108	572	696	14
47	231	106	536	107	464	695	13
48	337	105	643	108	357	693	12
49	442	106	751	107	249	692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	10
51	653	105	964	107	036	689	9
52	758	105	9.08 071	106	0.91 929	687	8
53	863	105	177	106	823	686	7
54	968	104	283	106	717	684	6
55	9.08 072	104	389	106	611	683	5
56	176	104	495	105	505	681	4
57	280	103	600	105	400	680	3
58	383	103	705	105	295	678	2
59	486	103	810	104	190	677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.			
1′	120	119	118
1″	2.0	2.0	2.0
2″	4.0	4.0	3.9
3″	6.0	6.0	5.9
4″	8.0	7.9	7.9
5″	10.0	9.9	9.8
6″	12.0	11.9	11.8
7″	14.0	13.9	13.8
8″	16.0	15.9	15.7
9″	18.0	17.9	17.7
1′	117	116	115
1″	2.0	1.9	1.9
2″	3.9	3.9	3.8
3″	5.9	5.8	5.8
4″	7.8	7.7	7.7
5″	9.8	9.7	9.6
6″	11.7	11.6	11.5
7″	13.7	13.5	13.4
8″	15.6	15.5	15.3
9″	17.6	17.4	17.3
1′	114	113	112
1″	1.9	1.9	1.9
2″	3.8	3.8	3.7
3″	5.7	5.7	5.6
4″	7.6	7.5	7.5
5″	9.5	9.4	9.3
6″	11.4	11.3	11.2
7″	13.3	13.2	13.1
8″	15.2	15.1	14.9
9″	17.1	17.0	16.8
1′	111	110	109
1″	1.9	1.8	1.8
2″	3.7	3.7	3.6
3″	5.6	5.5	5.5
4″	7.4	7.3	7.3
5″	9.3	9.2	9.1
6″	11.1	11.0	10.9
7″	13.0	12.8	12.7
8″	14.8	14.7	14.5
9″	16.7	16.5	16.4
1′	108	107	106
1″	1.8	1.8	1.8
2″	3.6	3.6	3.5
3″	5.4	5.4	5.3
4″	7.2	7.1	7.1
5″	9.0	8.9	8.8
6″	10.8	10.7	10.6
7″	12.6	12.5	12.4
8″	14.4	14.3	14.1
9″	16.2	16.1	15.9
1′	105	104	103
1″	1.8	1.7	1.7
2″	3.5	3.5	3.4
3″	5.3	5.2	5.2
4″	7.0	6.9	6.9
5″	8.8	8.7	8.6
6″	10.5	10.4	10.3
7″	12.3	12.1	12.0
8″	14.0	13.9	13.7
9″	15.8	15.6	15.5

83°

Logarithmic Functions

7°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	**60**
1	692	103	9.09 019	104	0.90 981	674	59
2	795	102	123	104	877	672	58
3	897	102	227	103	773	670	57
4	999	102	330	104	670	669	56
5	9.09 101	101	434	103	566	667	55
6	202	102	537	103	463	666	54
7	304	101	640	102	360	664	53
8	405	101	742	103	258	663	52
9	506	100	845	102	155	661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	**50**
11	707	100	9.10 049	101	0.89 951	658	49
12	807	100	150	102	850	656	48
13	907	99	252	101	748	655	47
14	9.10 006	100	353	101	647	653	46
15	106	99	454	101	546	651	45
16	205	99	555	101	445	650	44
17	304	98	656	100	344	648	43
18	402	99	756	100	244	647	42
19	501	98	856	100	144	645	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	**40**
21	697	98	9.11 056	99	0.88 944	642	39
22	795	98	155	99	845	640	38
23	893	97	254	99	746	638	37
24	990	97	353	99	647	637	36
25	9.11 087	97	452	99	548	635	35
26	184	97	551	98	449	633	34
27	281	96	649	98	351	632	33
28	377	97	747	98	253	630	32
29	474	96	845	98	155	629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	**30**
31	666	95	9.12 040	98	0.87 960	625	29
32	761	96	138	97	862	624	28
33	857	95	235	97	765	622	27
34	952	95	332	96	668	620	26
35	9.12 047	95	428	97	572	618	25
36	142	94	525	96	475	617	24
37	236	95	621	96	379	615	23
38	331	94	717	96	283	613	22
39	425	94	813	96	187	612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	**20**
41	612	94	9.13 004	95	0.86 996	608	19
42	706	93	099	95	901	607	18
43	799	93	194	95	806	605	17
44	892	93	289	95	711	603	16
45	985	93	384	94	616	601	15
46	9.13 078	93	478	95	522	600	14
47	171	92	573	94	427	598	13
48	263	92	667	94	333	596	12
49	355	92	761	93	239	595	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	**10**
51	539	91	948	93	052	591	9
52	630	92	9.14 041	93	0.85 959	589	8
53	722	91	134	93	866	588	7
54	813	91	227	93	773	586	6
55	904	90	320	92	680	584	5
56	994	91	412	92	588	582	4
57	9.14 085	90	504	93	496	581	3
58	175	91	597	91	403	579	2
59	266	90	688	92	312	577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

P. P.

1′	105	104	103
1″	1.8	1.7	1.7
2″	3.5	3.5	3.4
3″	5.3	5.2	5.2
4″	7.0	6.9	6.9
5″	8.8	8.7	8.6
6″	10.5	10.4	10.3
7″	12.3	12.1	12.0
8″	14.0	13.9	13.7
9″	15.8	15.6	15.5
1′	102	101	100
1″	1.7	1.7	1.7
2″	3.4	3.4	3.3
3″	5.1	5.1	5.0
4″	6.8	6.7	6.7
5″	8.5	8.4	8.3
6″	10.2	10.1	10.0
7″	11.9	11.8	11.7
8″	13.6	13.5	13.3
9″	15.3	15.2	15.0
1′	99	98	97
1″	1.7	1.6	1.6
2″	3.3	3.3	3.2
3″	5.0	4.9	4.9
4″	6.6	6.5	6.5
5″	8.3	8.2	8.1
6″	9.9	9.8	9.7
7″	11.6	11.4	11.3
8″	13.2	13.1	12.9
9″	14.9	14.7	14.6
1′	96	95	94
1″	1.6	1.6	1.6
2″	3.2	3.2	3.1
3″	4.8	4.8	4.7
4″	6.4	6.3	6.3
5″	8.0	7.9	7.8
6″	9.6	9.5	9.4
7″	11.2	11.1	11.0
8″	12.8	12.7	12.5
9″	14.4	14.3	14.1
1′	93	92	91
1″	1.6	1.5	1.5
2″	3.1	3.1	3.0
3″	4.7	4.6	4.6
4″	6.2	6.1	6.1
5″	7.8	7.7	7.6
6″	9.3	9.2	9.1
7″	10.9	10.7	10.6
8″	12.4	12.3	12.1
9″	14.0	13.8	13.7
1′	90	1	2
1″	1.5	0.02	0.03
2″	3.0	0.03	0.07
3″	4.5	0.05	0.10
4″	6.0	0.07	0.13
5″	7.5	0.08	0.17
6″	9.0	0.10	0.20
7″	10.5	0.12	0.23
8″	12.0	0.13	0.27
9″	13.5	0.15	0.30

82°

8°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	1	60
1	445	90	872	91	128	574	2	59
2	535	89	963	91	037	572	2	58
3	624	90	9.15 054	91	0.84 946	570	2	57
4	714	89	145	91	855	568	2	56
5	803	88	236	91	764	566	1	55
6	891	89	327	90	673	565	2	54
7	980	89	417	91	583	563	2	53
8	9.15 069	88	508	90	492	561	2	52
9	157	88	598	90	402	559	2	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	1	50
11	333	88	777	90	223	556	2	49
12	421	87	867	89	133	554	2	48
13	508	88	956	90	044	552	2	47
14	596	87	9.16 046	89	0.83 954	550	2	46
15	683	87	135	89	865	548	2	45
16	770	87	224	88	776	546	1	44
17	857	87	312	89	688	545	2	43
18	944	86	401	88	599	543	2	42
19	9.16 030	86	489	88	511	541	2	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	2	40
21	203	86	665	88	335	537	2	39
22	289	85	753	88	247	535	2	38
23	374	86	841	87	159	533	1	37
24	460	85	928	88	072	532	2	36
25	545	86	9.17 016	87	0.82 984	530	2	35
26	631	85	103	87	897	528	2	34
27	716	85	190	87	810	526	2	33
28	801	85	277	86	723	524	2	32
29	886	84	363	87	637	522	2	31
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	2	30
31	9.17 055	84	536	86	464	518	1	29
32	139	84	622	86	378	517	2	28
33	223	84	708	86	292	515	2	27
34	307	84	794	86	206	513	2	26
35	391	83	880	85	120	511	2	25
36	474	84	965	86	035	509	2	24
37	558	83	9.18 051	85	0.81 949	507	2	23
38	641	83	136	85	864	505	2	22
39	724	83	221	85	779	503	2	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	2	20
41	890	83	391	84	609	499	2	19
42	973	82	475	85	525	497	2	18
43	9.18 055	82	560	84	440	495	1	17
44	137	83	644	84	356	494	2	16
45	220	82	728	84	272	492	2	15
46	302	81	812	84	188	490	2	14
47	383	82	896	83	104	488	2	13
48	465	82	979	84	021	486	2	12
49	547	81	9.19 063	83	0.80 937	484	2	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	2	10
51	709	81	229	83	771	480	2	9
52	790	81	312	83	688	478	2	8
53	871	81	395	83	605	476	2	7
54	952	81	478	83	522	474	2	6
55	9.19 033	80	561	82	439	472	2	5
56	113	80	643	82	357	470	2	4
57	193	80	725	82	275	468	2	3
58	273	80	807	82	193	466	2	2
59	353	80	889	82	111	464	2	1
60	9.19 433		9.19 971		0.80 029	9.99 462		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	92	91	90
1″	1.5	1.5	1.5
2″	3.1	3.0	3.0
3″	4.6	4.6	4.5
4″	6.1	6.1	6.0
5″	7.7	7.6	7.5
6″	9.2	9.1	9.0
7″	10.7	10.6	10.5
8″	12.3	12.1	12.0
9″	13.8	13.7	13.5

1′	89	88	87
1″	1.5	1.5	1.5
2″	3.0	2.9	2.9
3″	4.5	4.4	4.4
4″	5.9	5.9	5.8
5″	7.4	7.3	7.3
6″	8.9	8.8	8.7
7″	10.4	10.3	10.2
8″	11.9	11.7	11.6
9″	13.4	13.2	13.1

1′	86	85
1″	1.4	1.4
2″	2.9	2.8
3″	4.3	4.3
4″	5.7	5.7
5″	7.2	7.1
6″	8.6	8.5
7″	10.0	9.9
8″	11.5	11.3
9″	12.9	12.8

1′	84	83
1″	1.4	1.4
2″	2.8	2.8
3″	4.2	4.2
4″	5.6	5.5
5″	7.0	6.9
6″	8.4	8.3
7″	9.8	9.7
8″	11.2	11.1
9″	12.6	12.5

1′	82	81
1″	1.4	1.4
2″	2.7	2.7
3″	4.1	4.1
4″	5.5	5.4
5″	6.8	6.8
6″	8.2	8.1
7″	9.6	9.5
8″	10.9	10.8
9″	12.3	12.2

1′	1	2	80
1″	0.02	0.03	1.3
2″	0.03	0.07	2.7
3″	0.05	0.10	4.0
4″	0.07	0.13	5.3
5″	0.08	0.17	6.7
6″	0.10	0.20	8.0
7″	0.12	0.23	9.3
8″	0.13	0.27	10.7
9″	0.15	0.30	12.0

81°

Logarithmic Functions

9°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	'
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	2	60
1	513	79	9.20 053	81	0.79 947	460	2	59
2	592	80	134	82	866	458	2	58
3	672	79	216	81	784	456	2	57
4	751	79	297	81	703	454	2	56
5	830	79	378	81	622	452	2	55
6	909	79	459	81	541	450	2	54
7	988	79	540	81	460	448	2	53
8	9.20 067	78	621	80	379	446	2	52
9	145	78	701	81	299	444	2	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	2	50
11	302	78	862	80	138	440	2	49
12	380	78	942	80	058	438	2	48
13	458	77	9.21 022	80	0.78 978	436	2	47
14	535	78	102	80	898	434	2	46
15	613	78	182	79	818	432	3	45
16	691	77	261	80	739	429	2	44
17	768	77	341	79	659	427	2	43
18	845	77	420	79	580	425	2	42
19	922	77	499	79	501	423	2	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	2	40
21	9.21 076	77	657	79	343	419	2	39
22	153	76	736	78	264	417	2	38
23	229	77	814	79	186	415	2	37
24	306	76	893	78	107	413	2	36
25	382	76	971	78	029	411	2	35
26	458	76	9.22 049	78	0.77 951	409	2	34
27	534	76	127	78	873	407	3	33
28	610	75	205	78	795	404	2	32
29	685	76	283	78	717	402	2	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	2	30
31	836	76	438	78	562	398	2	29
32	912	75	516	77	484	396	2	28
33	987	75	593	77	407	394	2	27
34	9.22 062	75	670	77	330	392	2	26
35	137	74	747	77	253	390	2	25
36	211	75	824	77	176	388	3	24
37	286	75	901	76	099	385	2	23
38	361	74	977	77	023	383	2	22
39	435	74	9.23 054	76	0.76 946	381	2	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	2	20
41	583	74	206	77	794	377	2	19
42	657	74	283	76	717	375	3	18
43	731	74	359	76	641	372	2	17
44	805	73	435	75	565	370	2	16
45	878	74	510	76	490	368	2	15
46	952	73	586	75	414	366	2	14
47	9.23 025	73	661	76	339	364	2	13
48	098	73	737	75	263	362	3	12
49	171	73	812	75	188	359	2	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	2	10
51	317	73	962	75	038	355	2	9
52	390	72	9.24 037	75	0.75 963	353	2	8
53	462	73	112	74	888	351	3	7
54	535	72	186	75	814	348	2	6
55	607	72	261	74	739	346	2	5
56	679	73	335	75	665	344	2	4
57	752	71	410	74	590	342	2	3
58	823	72	484	74	516	340	3	2
59	895	72	558	74	442	337	2	1
60	9.23 967		9.24 632		0.75 368	9.99 335		0
'	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

80°

P. P.

1′	82	81	80
1″	1.4	1.4	1.3
2″	2.7	2.7	2.7
3″	4.1	4.1	4.0
4″	5.5	5.4	5.3
5″	6.8	6.8	6.7
6″	8.2	8.1	8.0
7″	9.6	9.5	9.3
8″	10.9	10.8	10.7
9″	12.3	12.2	12.0

1′	79	78	77
1″	1.3	1.3	1.3
2″	2.6	2.6	2.6
3″	4.0	3.9	3.9
4″	5.3	5.2	5.1
5″	6.6	6.5	6.4
6″	7.9	7.8	7.7
7″	9.2	9.1	9.0
8″	10.5	10.4	10.3
9″	11.9	11.7	11.6

1′	76	75
1″	1.3	1.3
2″	2.5	2.5
3″	3.8	3.8
4″	5.1	5.0
5″	6.3	6.3
6″	7.6	7.5
7″	8.9	8.8
8″	10.1	10.0
9″	11.4	11.3

1′	74	73
1″	1.2	1.2
2″	2.5	2.4
3″	3.7	3.7
4″	4.9	4.9
5″	6.2	6.1
6″	7.4	7.3
7″	8.6	8.5
8″	9.9	9.7
9″	11.1	11.0

1′	72	71
1″	1.2	1.2
2″	2.4	2.4
3″	3.6	3.6
4″	4.8	4.7
5″	6.0	5.9
6″	7.2	7.1
7″	8.4	8.3
8″	9.6	9.5
9″	10.8	10.7

1′	2	3
1″	0.03	0.05
2″	0.07	0.10
3″	0.10	0.15
4″	0.13	0.20
5″	0.17	0.25
6″	0.20	0.30
7″	0.23	0.35
8″	0.27	0.40
9″	0.30	0.45

10°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.23 967	72	9.24 632	74	0.75 368	9.99 335	2	60
1	9.24 039	71	706	73	294	333	2	59
2	110	71	779	74	221	331	3	58
3	181	72	853	73	147	328	2	57
4	253	71	926	74	074	326	2	56
5	324	71	9.25 000	73	000	324	2	55
6	395	71	073	73	0.74 927	322	3	54
7	466	70	146	73	854	319	2	53
8	536	71	219	73	781	317	2	52
9	607	70	292	73	708	315	2	51
10	9.24 677	71	9.25 365	72	0.74 635	9.99 313	3	50
11	748	70	437	73	563	310	2	49
12	818	70	510	72	490	308	2	48
13	888	70	582	73	418	306	2	47
14	958	70	655	72	345	304	3	46
15	9.25 028	70	727	72	273	301	2	45
16	098	70	799	72	201	299	2	44
17	168	69	871	72	129	297	3	43
18	237	70	943	72	057	294	2	42
19	307	69	9.26 015	71	0.73 985	292	2	41
20	9.25 376	69	9.26 086	72	0.73 914	9.99 290	2	40
21	445	69	158	71	842	288	3	39
22	514	69	229	72	771	285	2	38
23	583	69	301	71	699	283	2	37
24	652	69	372	71	628	281	3	36
25	721	69	443	71	557	278	2	35
26	790	68	514	71	486	276	2	34
27	858	69	585	70	415	274	3	33
28	927	68	655	71	345	271	2	32
29	995	68	726	71	274	269	2	31
30	9.26 063	68	9.26 797	70	0.73 203	9.99 267	3	30
31	131	68	867	70	133	264	2	29
32	199	68	937	71	063	262	2	28
33	267	68	9.27 008	70	0.72 992	260	3	27
34	335	68	078	70	922	257	2	26
35	403	67	148	70	852	255	3	25
36	470	68	218	70	782	252	2	24
37	538	67	288	69	712	250	2	23
38	605	67	357	70	643	248	3	22
39	672	67	427	69	573	245	2	21
40	9.26 739	67	9.27 496	70	0.72 504	9.99 243	2	20
41	806	67	566	69	434	241	3	19
42	873	67	635	69	365	238	2	18
43	940	67	704	69	296	236	3	17
44	9.27 007	66	773	69	227	233	2	16
45	073	67	842	69	158	231	2	15
46	140	66	911	69	089	229	3	14
47	206	67	980	69	020	226	2	13
48	273	66	9.28 049	68	0.71 951	224	3	12
49	339	66	117	69	883	221	2	11
50	9.27 405	66	9.28 186	68	0.71 814	9.99 219	2	10
51	471	66	254	69	746	217	3	9
52	537	65	323	68	677	214	2	8
53	602	66	391	68	609	212	3	7
54	668	66	459	68	541	209	2	6
55	734	65	527	68	473	207	3	5
56	799	65	595	67	405	204	2	4
57	864	66	662	68	338	202	2	3
58	930	65	730	68	270	200	3	2
59	995	65	798	67	202	197	2	1
60	9.28 060		9.28 865		0.71 135	9.99 195		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.		
1′	74	73
1″	1.2	1.2
2″	2.5	2.4
3″	3.7	3.7
4″	4.9	4.9
5″	6.2	6.1
6″	7.4	7.3
7″	8.6	8.5
8″	9.9	9.7
9″	11.1	11.0
1′	72	71
1″	1.2	1.2
2″	2.4	2.4
3″	3.6	3.6
4″	4.8	4.7
5″	6.0	5.9
6″	7.2	7.1
7″	8.4	8.3
8″	9.6	9.5
9″	10.8	10.7
1′	70	69
1″	1.2	1.2
2″	2.3	2.3
3″	3.5	3.5
4″	4.7	4.6
5″	5.8	5.8
6″	7.0	6.9
7″	8.2	8.1
8″	9.3	9.2
9″	10.5	10.4
1′	68	67
1″	1.1	1.1
2″	2.3	2.2
3″	3.4	3.4
4″	4.5	4.5
5″	5.7	5.6
6″	6.8	6.7
7″	7.9	7.8
8″	9.1	8.9
9″	10.2	10.1
1′	66	65
1″	1.1	1.1
2″	2.2	2.2
3″	3.3	3.3
4″	4.4	4.3
5″	5.5	5.4
6″	6.6	6.5
7″	7.7	7.6
8″	8.8	8.7
9″	9.9	9.8
1′	2	3
1″	0.03	0.05
2″	0.07	0.10
3″	0.10	0.15
4″	0.13	0.20
5″	0.17	0.25
6″	0.20	0.30
7″	0.23	0.35
8″	0.27	0.40
9″	0.30	0.45

79°

Logarithmic Functions

11°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.28 $06\underline{0}$	65	9.28 865	68	0.71 $13\bar{5}$	9.99 $19\bar{5}$	3	60
1	$12\bar{5}$	65	933	67	067	192	2	59
2	190	64	9.29 000	67	000	190	3	58
3	254	65	067	67	0.70 933	$18\underline{7}$	2	57
4	319	65	134	67	866	$18\bar{5}$	3	56
5	384	64	201	67	799	182	2	55
6	448	64	$26\underline{8}$	67	732	180	3	54
7	512	65	$33\bar{5}$	67	665	$17\underline{7}$	2	53
8	577	64	402	66	598	$17\bar{5}$	3	52
9	$64\underline{1}$	64	$46\underline{8}$	67	532	172	2	51
10	9.28 $70\bar{5}$	64	9.29 $53\bar{5}$	66	0.70 465	9.99 170	3	**50**
11	769	64	601	67	399	$16\underline{7}$	2	49
12	833	63	668	66	332	$16\bar{5}$	3	48
13	896	64	734	66	266	162	2	47
14	960	64	800	66	200	160	3	46
15	9.29 024	63	866	66	134	157	2	45
16	087	63	932	66	068	$15\bar{5}$	3	44
17	150	64	998	66	002	152	2	43
18	214	63	9.30 064	66	0.69 936	150	3	42
19	277	63	130	65	$87\underline{0}$	$14\underline{7}$	2	41
20	9.29 340	63	9.30 195	66	0.69 $80\bar{5}$	9.99 $14\bar{5}$	3	**40**
21	403	63	261	65	739	142	2	39
22	466	63	326	65	674	140	3	38
23	529	62	391	66	609	$13\underline{7}$	2	37
24	591	63	457	65	543	$13\bar{5}$	3	36
25	654	62	522	65	478	132	2	35
26	716	63	587	65	413	130	3	34
27	779	62	652	65	348	127	3	33
28	841	62	717	65	283	124	2	32
29	903	63	782	64	218	122	3	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	2	**30**
31	9.30 028	62	911	64	089	117	3	29
32	090	61	975	65	$02\bar{5}$	114	2	28
33	151	62	9.31 040	64	0.68 960	112	3	27
34	$21\underline{3}$	62	104	64	896	109	3	26
35	$27\bar{5}$	61	168	65	832	106	2	25
36	336	62	233	64	767	104	3	24
37	398	61	297	64	703	101	2	23
38	459	62	$36\underline{1}$	64	639	099	3	22
39	521	61	$42\bar{5}$	64	575	096	3	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	2	**20**
41	643	61	552	64	448	091	3	19
42	704	61	616	63	384	088	2	18
43	765	61	679	64	321	086	3	17
44	826	61	743	63	257	083	3	16
45	887	60	806	64	194	080	2	15
46	947	61	870	63	130	078	3	14
47	9.31 008	60	933	63	067	075	3	13
48	068	61	996	63	004	072	2	12
49	129	60	9.32 059	63	0.67 941	070	3	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	**10**
51	250	60	185	63	$81\bar{5}$	064	2	9
52	310	60	248	63	752	062	3	8
53	370	60	311	62	689	059	3	7
54	430	60	373	63	627	056	2	6
55	490	59	436	62	564	054	3	5
56	549	60	498	63	502	051	3	4
57	609	60	561	62	439	048	2	3
58	669	59	623	62	$37\underline{7}$	046	3	2
59	728	60	685	62	$31\bar{5}$	043	3	1
60	9.31 788		9.32 747		0.67 253	9.99 040		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.		
1′	68	67
1″	1.1	1.1
2″	2.3	2.2
3″	3.4	3.4
4″	4.5	4.5
5″	5.7	5.6
6″	6.8	6.7
7″	7.9	7.8
8″	9.1	8.9
9″	10.2	10.1
1′	66	65
1″	1.1	1.1
2″	2.2	2.2
3″	3.3	3.3
4″	4.4	4.3
5″	5.5	5.4
6″	6.6	6.5
7″	7.7	7.6
8″	8.8	8.7
9″	9.9	9.8
1′	64	63
1″	1.1	1.1
2″	2.1	2.1
3″	3.2	3.2
4″	4.3	4.2
5″	5.3	5.3
6″	6.4	6.3
7″	7.5	7.4
8″	8.5	8.4
9″	9.6	9.5
1′	62	61
1″	1.0	1.0
2″	2.1	2.0
3″	3.1	3.1
4″	4.1	4.1
5″	5.2	5.1
6″	6.2	6.1
7″	7.2	7.1
8″	8.3	8.1
9″	9.3	9.2
1′	60	59
1″	1.0	1.0
2″	2.0	2.0
3″	3.0	3.0
4″	4.0	3.9
5″	5.0	4.9
6″	6.0	5.9
7″	7.0	6.9
8″	8.0	7.9
9″	9.0	8.9
1′	2	3
1″	0.03	0.05
2″	0.07	0.10
3″	0.10	0.15
4″	0.13	0.20
5″	0.17	0.25
6″	0.20	0.30
7″	0.23	0.35
8″	0.27	0.40
9″	0.30	0.45

78°

12°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	'
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	**60**
1	847	60	810	62	190	038	3	59
2	907	59	872	61	128	035	3	58
3	966	59	933	62	067	032	2	57
4	9.32 02$\bar{5}$	59	995	62	00$\bar{5}$	030	3	56
5	084	59	9.33 057	62	0.66 943	027	3	55
6	143	59	119	61	881	024	2	54
7	202	59	180	62	820	022	3	53
8	261	58	242	61	758	019	3	52
9	319	59	303	62	697	016	3	51
10	9.32 378	59	9.33 36$\bar{5}$	61	0.66 635	9.99 013	2	**50**
11	437	58	426	61	574	011	3	49
12	495	58	487	61	513	008	3	48
13	553	59	548	61	452	005	3	47
14	612	58	609	61	391	002	2	46
15	670	58	670	61	330	000	3	45
16	728	58	731	61	269	9.98 997	3	44
17	786	58	792	61	208	994	3	43
18	844	58	853	60	147	991	2	42
19	902	58	913	61	087	989	3	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	3	**40**
21	9.33 018	57	9.34 034	61	0.65 966	983	3	39
22	075	58	09$\bar{5}$	60	905	980	2	38
23	133	57	155	60	84$\bar{5}$	978	3	37
24	190	58	215	61	78$\bar{5}$	97$\bar{5}$	3	36
25	248	57	276	60	724	972	3	35
26	305	57	336	60	664	969	2	34
27	362	58	396	60	604	967	3	33
28	420	57	456	60	544	964	3	32
29	477	57	516	60	484	961	3	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	3	**30**
31	591	56	635	60	36$\bar{5}$	955	2	29
32	647	57	69$\bar{5}$	60	305	953	3	28
33	704	57	75$\bar{5}$	59	245	950	3	27
34	761	57	814	60	186	947	3	26
35	818	56	874	59	126	944	3	25
36	874	57	933	59	067	941	3	24
37	931	56	992	59	008	938	2	23
38	987	56	9.35 051	60	0.64 949	936	3	22
39	9.34 043	57	111	59	889	933	3	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	**20**
41	156	56	229	59	771	927	3	19
42	212	56	288	59	712	924	3	18
43	268	56	347	58	653	921	2	17
44	324	56	405	59	59$\bar{5}$	919	3	16
45	380	56	464	59	536	916	3	15
46	436	55	523	58	477	913	3	14
47	491	56	581	59	419	910	3	13
48	547	55	640	58	360	907	3	12
49	602	56	698	59	302	904	3	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	**10**
51	713	56	81$\bar{5}$	58	185	898	2	9
52	769	55	873	58	127	896	3	8
53	824	55	931	58	069	893	3	7
54	879	55	989	58	011	890	3	6
55	934	55	9.36 047	58	0.63 953	887	3	5
56	989	55	105	58	89$\bar{5}$	884	3	4
57	9.35 044	55	163	58	837	881	3	3
58	099	55	221	58	779	878	3	2
59	154	55	279	57	721	875	3	1
60	9.35 209		9.36 336		0.63 664	9.98 872		0
'	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

P. P.

1'	63	62
1″	1.1	1.0
2″	2.1	2.1
3″	3.2	3.1
4″	4.2	4.1
5″	5.3	5.2
6″	6.3	6.2
7″	7.4	7.2
8″	8.4	8.3
9″	9.5	9.3
1'	61	60
1″	1.0	1.0
2″	2.0	2.0
3″	3.1	3.0
4″	4.1	4.0
5″	5.1	5.0
6″	6.1	6.0
7″	7.1	7.0
8″	8.1	8.0
9″	9.2	9.0
1'	59	58
1″	1.0	1.0
2″	2.0	1.9
3″	3.0	2.9
4″	3.9	3.9
5″	4.9	4.8
6″	5.9	5.8
7″	6.9	6.8
8″	7.9	7.7
9″	8.9	8.7
1'	57	56
1″	1.0	0.9
2″	1.9	1.9
3″	2.9	2.8
4″	3.8	3.7
5″	4.8	4.7
6″	5.7	5.6
7″	6.7	6.5
8″	7.6	7.5
9″	8.6	8.4
1'	55	
1″	0.9	
2″	1.8	
3″	2.8	
4″	3.7	
5″	4.6	
6″	5.5	
7″	6.4	
8″	7.3	
9″	8.3	
1'	2	3
1″	0.03	0.05
2″	0.07	0.10
3″	0.10	0.15
4″	0.13	0.20
5″	0.17	0.25
6″	0.20	0.30
7″	0.23	0.35
8″	0.27	0.40
9″	0.30	0.45

77°

13°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.35 209	54	9.36 336	58	0.63 664	9.98 872	3	**60**
1	263	55	394	58	606	869	2	59
2	318	55	452	57	548	867	3	58
3	373	54	509	57	491	864	3	57
4	427	54	566	58	434	861	3	56
5	481	55	624	57	376	858	3	55
6	536	54	681	57	319	855̅	3	54
7	590	54	738	57	262	852	3	53
8	644	54	795	57	205̅	849	3	52
9	698	54	852	57	148	846	3	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	**50**
11	806	54	966	57	034	840	3	49
12	860	54	9.37 023	57	0.62 977	837	3	48
13	914	54	080	57	920	834	3	47
14	968	54	137	56	863	831	3	46
15	9.36 022	53	193	57	807	828	3	45
16	075	54	250	56	750	825	3	44
17	129	53	306	57	694	822	3	43
18	182	54	363	56	637	819	3	42
19	236	53	419	57	581	816	3	41
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	3	**40**
21	342	53	532	56	468	810	3	39
22	395	54	588	56	412	807	3	38
23	449	53	644	56	356	804	3	37
24	502	53	700	56	300	801	3	36
25	555̅	53	756	56	244	798	3	35
26	608	52	812	56	188	795	3	34
27	660	53	868	56	132	792	3	33
28	713	53	924	56	076	789	3	32
29	766	53	980	55	020	786	3	31
30	9.36 819	52	9.38 035	56	0.61 965̅	9.98 783	3	**30**
31	871	53	091	56	909	780	3	29
32	924	52	147	55	853	777	3	28
33	976	52	202	55	798	774	3	27
34	9.37 028	53	257	56	743	771	3	26
35	081	52	313	55	687	768	3	25
36	133	52	368	55	632	765̅	3	24
37	185	52	423	56	577	762	3	23
38	237	52	479	55	521	759	3	22
39	289	52	534	55	466	756	3	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	**20**
41	393	52	644	55	356	750	4	19
42	445	52	699	55	301	746	3	18
43	497	52	754	54	246	743	3	17
44	549	51	808	55	192	740	3	16
45	600	52	863	55	137	737	3	15
46	652	51	918	54	082	734	3	14
47	703	52	972	55	028	731	3	13
48	755̅	51	9.39 027	55	0.60 973	728	3	12
49	806	52	082	54	918	725̅	3	11
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	3	**10**
51	909	51	190	55	810	719	4	9
52	960	51	245̅	54	755	715	3	8
53	9.38 011	51	299	54	701	712	3	7
54	062	51	353	54	647	709	3	6
55	113	51	407	54	593	706	3	5
56	164	51	461	54	539	703	3	4
57	215	51	515	54	485̅	700	3	3
58	266	51	569	54	431	697	3	2
59	317	51	623	54	377	694	4	1
60	9.38 368		9.39 677		0.60 323	9.98 690		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.		
1′	58	57
1″	1.0	1.0
2″	1.9	1.9
3″	2.9	2.9
4″	3.9	3.8
5″	4.8	4.8
6″	5.8	5.7
7″	6.8	6.7
8″	7.7	7.6
9″	8.7	8.6
1′	56	55
1″	0.9	0.9
2″	1.9	1.8
3″	2.8	2.8
4″	3.7	3.7
5″	4.7	4.6
6″	5.6	5.5
7″	6.5	6.4
8″	7.5	7.3
9″	8.4	8.3
1′	54	53
1″	0.9	0.9
2″	1.8	1.8
3″	2.7	2.7
4″	3.6	3.5
5″	4.5	4.4
6″	5.4	5.3
7″	6.3	6.2
8″	7.2	7.1
9″	8.1	8.0
1′	52	51
1″	0.9	0.9
2″	1.7	1.7
3″	2.6	2.6
4″	3.5	3.4
5″	4.3	4.3
6″	5.2	5.1
7″	6.1	6.0
8″	6.9	6.8
9″	7.8	7.7
1′	2	
1″	0.03	
2″	0.07	
3″	0.10	
4″	0.13	
5″	0.17	
6″	0.20	
7″	0.23	
8″	0.27	
9″	0.30	
1′	3	4
1″	0.05	0.07
2″	0.10	0.13
3″	0.15	0.20
4″	0.20	0.27
5″	0.25	0.33
6″	0.30	0.40
7″	0.35	0.47
8″	0.40	0.53
9″	0.45	0.60

76°

14°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	60
1	418	51	731	54	269	687	3	59
2	469	50	785	53	215	684	3	58
3	519	51	838	54	162	681	3	57
4	570	50	892	53	108	678	3	56
5	620	50	945	54	055	675	4	55
6	670	51	999	53	001	671	3	54
7	721	50	9.40 052	54	0.59 948	668	3	53
8	771	50	106	53	894	665	3	52
9	821	50	159	53	841	662	3	51
10	9.38 871	50	9.40 212	54	0.59 788	9.98 659	3	50
11	921	50	266	53	734	656	4	49
12	971	50	319	53	681	652	3	48
13	9.39 021	50	372	53	628	649	3	47
14	071	50	425	53	575	646	3	46
15	121	49	478	53	522	643	3	45
16	170	50	531	53	469	640	4	44
17	220	50	584	52	416	636	3	43
18	270	49	636	53	364	633	3	42
19	319	50	689	53	311	630	3	41
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	4	40
21	418	49	795	52	205	623	3	39
22	467	50	847	53	153	620	3	38
23	517	49	900	52	100	617	3	37
24	566	49	952	53	048	614	4	36
25	615	49	9.41 005	52	0.58 995	610	3	35
26	664	49	057	52	943	607	3	34
27	713	49	109	52	891	604	3	33
28	762	49	161	53	839	601	4	32
29	811	49	214	52	786	597	3	31
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	30
31	909	49	318	52	682	591	3	29
32	958	48	370	52	630	588	4	28
33	9.40 006	49	422	52	578	584	3	27
34	055	48	474	52	526	581	3	26
35	103	49	526	52	474	578	4	25
36	152	48	578	51	422	574	3	24
37	200	49	629	52	371	571	3	23
38	249	48	681	52	319	568	3	22
39	297	49	733	51	267	565	4	21
40	9.40 346	48	9.41 784	52	0.58 216	9.98 561	3	20
41	394	48	836	51	164	558	3	19
42	442	48	887	52	113	555	4	18
43	490	48	939	51	061	551	3	17
44	538	48	990	51	010	548	3	16
45	586	48	9.42 041	52	0.57 959	545	4	15
46	634	48	093	51	907	541	3	14
47	682	48	144	51	856	538	3	13
48	730	48	195	51	805	535	4	12
49	778	47	246	51	754	531	3	11
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	10
51	873	48	348	51	652	525	4	9
52	921	47	399	51	601	521	3	8
53	968	48	450	51	550	518	3	7
54	9.41 016	47	501	51	499	515	4	6
55	063	48	552	51	448	511	3	5
56	111	47	603	50	397	508	3	4
57	158	47	653	51	347	505	4	3
58	205	47	704	51	296	501	3	2
59	252	48	755	50	245	498	4	1
60	9.41 300		9.42 805		0.57 195	9.98 494		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	54	53
1″	0.9	0.9
2″	1.8	1.8
3″	2.7	2.7
4″	3.6	3.5
5″	4.5	4.4
6″	5.4	5.3
7″	6.3	6.2
8″	7.2	7.1
9″	8.1	8.0

1′	52	51
1″	0.9	0.9
2″	1.7	1.7
3″	2.6	2.6
4″	3.5	3.4
5″	4.3	4.3
6″	5.2	5.1
7″	6.1	6.0
8″	6.9	6.8
9″	7.8	7.7

1′	50	49
1″	0.8	0.8
2″	1.7	1.6
3″	2.5	2.5
4″	3.3	3.3
5″	4.2	4.1
6″	5.0	4.9
7″	5.8	5.7
8″	6.7	6.5
9″	7.5	7.4

1′	48
1″	0.8
2″	1.6
3″	2.4
4″	3.2
5″	4.0
6″	4.8
7″	5.6
8″	6.4
9″	7.2

1′	47
1″	0.8
2″	1.6
3″	2.4
4″	3.1
5″	3.9
6″	4.7
7″	5.5
8″	6.3
9″	7.1

1′	3	4
1″	0.05	0.07
2″	0.10	0.13
3″	0.15	0.20
4″	0.20	0.27
5″	0.25	0.33
6″	0.30	0.40
7″	0.35	0.47
8″	0.40	0.53
9″	0.45	0.60

75°

Logarithmic Functions

15°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	'
0	9.41 300	47	9.42 805	51	0.57 195̅	9.98 494	3	**60**
1	347	47	856	50	144	491	3	59
2	394	47	906	51	094	488	4	58
3	441	47	957	50	043	484	3	57
4	488̲	47	9.43 007	50	0.56 993	481	4	56
5	535̅	47	057	51	943	477	3	55
6	582	46	108	50	892	474	3	54
7	628	47	158	50	842	471	4	53
8	675	47	208	50	792	467	3	52
9	722	46	258	50	742	464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	**50**
11	815	46	358	50	642	457	4	49
12	861	47	408	50	592	453	3	48
13	908	46	458	50	542	450	3	47
14	954	47	508	50	492	447	4	46
15	9.42 001	46	558	49	442	443	3	45
16	047	46	607	50	393	440	4	44
17	093	47	657	50	343	436	3	43
18	140	46	707	49	293	433	4	42
19	186	46	756	50	244	429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	**40**
21	278	46	855̲	50	145̅	422	3	39
22	324	46	905̅	49	095	419	4	38
23	370	46	954	50	046	415	3	37
24	416	45	9.44 004	49	0.55 996	412	3	36
25	461	46	053	49	947	409	4	35
26	507	46	102	49	898	405	3	34
27	553	46	151	50	849	402	4	33
28	599	45	201	49	799	398̲	3	32
29	644	46	250	49	750	395̅	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	**30**
31	735	46	348	49	652	388	4	29
32	781	45	397	49	603	384	3	28
33	826	46	446̲	49	554	381	4	27
34	872	45	495̅	49	505	377	4	26
35	917	45	544	48	456	373	3	25
36	962	46	592	49	408	370	4	24
37	9.43 008	45	641	49	359	366	3	23
38	053	45	690	48	310	363	4	22
39	098	45	738	49	262	359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	**20**
41	188	45	836	48	164	352	3	19
42	233	45	884	49	116	349	4	18
43	278	45	933	48	067	345	3	17
44	323	44	981	48	019	342	4	16
45	367	45	9.45 029	49	0.54 971	338	4	15
46	412	45	078	48	922	334	3	14
47	457	45	126	48	874	331	4	13
48	502	44	174	48	826	327	3	12
49	546	45	222	49	778	324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	**10**
51	635	45	319	48	681	317	4	9
52	680	44	367̲	48	633	313	4	8
53	724	45	415̅	48	585	309	3	7
54	769	44	463	48	537	306	4	6
55	813	44	511	48	489	302	3	5
56	857	44	559	47	441	299	4	4
57	901	45	606	48	394	295	4	3
58	946	44	654	48	346	291	3	2
59	990	44	702	48	298	288	4	1
60	9.44 034		9.45 750		0.54 250	9.98 284		**0**
'	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

P. P.

1'	51	50
1″	0.9	0.8
2″	1.7	1.7
3″	2.6	2.5
4″	3.4	3.3
5″	4.3	4.2
6″	5.1	5.0
7″	6.0	5.8
8″	6.8	6.7
9″	7.7	7.5

1'	49	48
1″	0.8	0.8
2″	1.6	1.6
3″	2.5	2.4
4″	3.3	3.2
5″	4.1	4.0
6″	4.9	4.8
7″	5.7	5.6
8″	6.5	6.4
9″	7.4	7.2

1'	47	46
1″	0.8	0.8
2″	1.6	1.5
3″	2.4	2.3
4″	3.1	3.1
5″	3.9	3.8
6″	4.7	4.6
7″	5.5	5.4
8″	6.3	6.1
9″	7.1	6.9

1'	45
1″	0.8
2″	1.5
3″	2.3
4″	3.0
5″	3.8
6″	4.5
7″	5.3
8″	6.0
9″	6.8

1'	44
1″	0.7
2″	1.5
3″	2.2
4″	2.9
5″	3.7
6″	4.4
7″	5.1
8″	5.9
9″	6.6

1'	3	4
1″	0.05	0.07
2″	0.10	0.13
3″	0.15	0.20
4″	0.20	0.27
5″	0.25	0.33
6″	0.30	0.40
7″	0.35	0.47
8″	0.40	0.53
9″	0.45	0.60

74°

16°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.44 034	44	9.45 750	47	0.54 250	9.98 284	3	**60**
1	078	44	797	48	203	281	4	59
2	122	44	845	47	155	277	4	58
3	166	44	892	48	108	273	3	57
4	210	43	940	47	060	270	4	56
5	253	44	987	48	013	266	4	55
6	297	44	9.46 035	47	0.53 965	262	3	54
7	341	44	082	48	918	259	4	53
8	385	43	130	47	870	255	4	52
9	428	44	177	47	823	251	3	51
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	4	**50**
11	516	43	271	48	729	244	4	49
12	559	43	319	47	681	240	3	48
13	602	44	366	47	634	237	4	47
14	646	43	413	47	587	233	4	46
15	689	44	460	47	540	229	3	45
16	733	43	507	47	493	226	4	44
17	776	43	554	47	446	222	4	43
18	819	43	601	47	399	218	3	42
19	862	43	648	46	352	215	4	41
20	9.44 905	43	9.46 694	47	0.53 306	9.98 211	4	**40**
21	948	44	741	47	259	207	3	39
22	992	43	788	47	212	204	4	38
23	9.45 035	42	835	46	165	200	4	37
24	077	43	881	47	119	196	4	36
25	120	43	928	47	072	192	3	35
26	163	43	975	46	025	189	4	34
27	206	43	9.47 021	47	0.52 979	185	4	33
28	249	43	068	46	932	181	4	32
29	292	42	114	46	886	177	3	31
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	4	**30**
31	377	42	207	46	793	170	4	29
32	419	43	253	46	747	166	4	28
33	462	42	299	47	701	162	3	27
34	504	43	346	46	654	159	4	26
35	547	42	392	46	608	155	4	25
36	589	43	438	46	562	151	4	24
37	632	42	484	46	516	147	3	23
38	674	42	530	46	470	144	4	22
39	716	42	576	46	424	140	4	21
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	**20**
41	801	42	668	46	332	132	3	19
42	843	42	714	46	286	129	4	18
43	885	42	760	46	240	125	4	17
44	927	42	806	46	194	121	4	16
45	969	42	852	45	148	117	4	15
46	9.46 011	42	897	46	103	113	3	14
47	053	42	943	46	057	110	4	13
48	095	41	989	46	011	106	4	12
49	136	42	9.48 035	45	0.51 965	102	4	11
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	**10**
51	220	42	126	45	874	094	4	9
52	262	41	171	46	829	090	3	8
53	303	42	217	45	783	087	4	7
54	345	41	262	45	738	083	4	6
55	386	42	307	46	693	079	4	5
56	428	41	353	45	647	075	4	4
57	469	42	398	45	602	071	4	3
58	511	41	443	46	557	067	4	2
59	552	42	489	45	511	063	3	1
60	9.46 594		9.48 534		0.51 466	9.98 060		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	48	47
1″	0.8	0.8
2″	1.6	1.6
3″	2.4	2.4
4″	3.2	3.1
5″	4.0	3.9
6″	4.8	4.7
7″	5.6	5.5
8″	6.4	6.3
9″	7.2	7.1
1′	46	45
1″	0.8	0.8
2″	1.5	1.5
3″	2.3	2.3
4″	3.1	3.0
5″	3.8	3.8
6″	4.6	4.5
7″	5.4	5.3
8″	6.1	6.0
9″	6.9	6.8
1′	44	43
1″	0.7	0.7
2″	1.5	1.4
3″	2.2	2.2
4″	2.9	2.9
5″	3.7	3.6
6″	4.4	4.3
7″	5.1	5.0
8″	5.9	5.7
9″	6.6	6.5
1′	42	
1″	0.7	
2″	1.4	
3″	2.1	
4″	2.8	
5″	3.5	
6″	4.2	
7″	4.9	
8″	5.6	
9″	6.3	
1′	41	
1″	0.7	
2″	1.4	
3″	2.1	
4″	2.7	
5″	3.4	
6″	4.1	
7″	4.8	
8″	5.5	
9″	6.2	
1′	3	4
1″	0.05	0.07
2″	0.10	0.13
3″	0.15	0.20
4″	0.20	0.27
5″	0.25	0.33
6″	0.30	0.40
7″	0.35	0.47
8″	0.40	0.53
9″	0.45	0.60

73°

17°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60
1	635	41	579	45	421	056	4	59
2	676	41	624	45	376	052	4	58
3	717	41	669	45	331	048	4	57
4	758	42	714	45	286	044	4	56
5	800	41	759	45	241	040	4	55
6	841	41	804	45	196	036	4	54
7	882	41	849	45	151	032	3	53
8	923	41	894	45	106	029	4	52
9	964	41	939	45	061	025	4	51
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50
11	045	41	9.49 029	44	0.50 971	017	4	49
12	086	41	073	45	927	013	4	48
13	127	41	118	45	882	009	4	47
14	168	41	163	44	837	005	4	46
15	209	40	207	45	793	001	4	45
16	249	41	252	44	748	9.97 997	4	44
17	290	40	296	45	704	993	4	43
18	330	41	341	44	659	989	3	42
19	371	40	385	45	615	986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	40
21	452	40	474	45	526	978	4	39
22	492	41	519	44	481	974	4	38
23	533	40	563	44	437	970	4	37
24	573	40	607	45	393	966	4	36
25	613	41	652	44	348	962	4	35
26	654	40	696	44	304	958	4	34
27	694	40	740	44	260	954	4	33
28	734	40	784	44	216	950	4	32
29	774	40	828	44	172	946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30
31	854	40	916	44	084	938	4	29
32	894	40	960	44	040	934	4	28
33	934	40	9.50 004	44	0.49 996	930	4	27
34	974	40	048	44	952	926	4	26
35	9.48 014	40	092	44	908	922	4	25
36	054	40	136	44	864	918	4	24
37	094	39	180	43	820	914	4	23
38	133	40	223	44	777	910	4	22
39	173	40	267	44	733	906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20
41	252	40	355	43	645	898	4	19
42	292	40	398	44	602	894	4	18
43	332	39	442	43	558	890	4	17
44	371	40	485	44	515	886	4	16
45	411	39	529	43	471	882	4	15
46	450	40	572	44	428	878	4	14
47	490	39	616	43	384	874	4	13
48	529	39	659	44	341	870	4	12
49	568	39	703	43	297	866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	10
51	647	39	789	44	211	857	4	9
52	686	39	833	43	167	853	4	8
53	725	39	876	43	124	849	4	7
54	764	39	919	43	081	845	4	6
55	803	39	962	43	038	841	4	5
56	842	39	9.51 005	43	0.48 995	837	4	4
57	881	39	048	44	952	833	4	3
58	920	39	092	43	908	829	4	2
59	959	39	135	43	865	825	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.		
1′	45	44
1″	0.8	0.7
2″	1.5	1.5
3″	2.3	2.2
4″	3.0	2.9
5″	3.8	3.7
6″	4.5	4.4
7″	5.3	5.1
8″	6.0	5.9
9″	6.8	6.6
1′	43	42
1″	0.7	0.7
2″	1.4	1.4
3″	2.2	2.1
4″	2.9	2.8
5″	3.6	3.5
6″	4.3	4.2
7″	5.0	4.9
8″	5.7	5.6
9″	6.5	6.3
1′	41	40
1″	0.7	0.7
2″	1.4	1.3
3″	2.1	2.0
4″	2.7	2.7
5″	3.4	3.3
6″	4.1	4.0
7″	4.8	4.7
8″	5.5	5.3
9″	6.2	6.0
1′	39	
1″	0.7	
2″	1.3	
3″	2.0	
4″	2.6	
5″	3.3	
6″	3.9	
7″	4.6	
8″	5.2	
9″	5.9	
1′	3	
1″	0.05	
2″	0.10	
3″	0.15	
4″	0.20	
5″	0.25	
6″	0.30	
7″	0.35	
8″	0.40	
9″	0.45	
1′	4	5
1″	0.07	0.08
2″	0.13	0.17
3″	0.20	0.25
4″	0.27	0.33
5″	0.33	0.42
6″	0.40	0.50
7″	0.47	0.58
8″	0.53	0.67
9″	0.60	0.75

72°

18°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	**60**
1	9.49 037	39	221	43	779	817	5	59
2	076̲	39	264	42	736	812	4	58
3	115̅	38	306	43	694	808	4	57
4	153	39	349	43	651	804	4	56
5	192	39	392	43	608	800	4	55
6	231	38	435̅	43	565	796	4	54
7	269	39	478	42	522	792	4	53
8	308	39	520	43	480	788	4	52
9	347	38	563	43	437	784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	**50**
11	424	38	648	43	352	775	4	49
12	462	38	691	43	309	771	4	48
13	500	39	734	42	266	767	4	47
14	539	38	776	43	224	763	4	46
15	577	38	819	42	181	759	5	45
16	615	39	861	42	139	754	4	44
17	654	38	903	43	097	750	4	43
18	692	38	946	42	054	746	4	42
19	730	38	988	43	012	742	4	41
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	**40**
21	806	38	073	42	927̲	734	5	39
22	844	38	115	42	885̅	729	4	38
23	882	38	157	43	843	725	4	37
24	920	38	200	42	800	721	4	36
25	958	38	242	42	758	717	4	35
26	996	38	284	42	716	713	5	34
27	9.50 034	38	326	42	674	708	4	33
28	072	38	368	42	632	704	4	32
29	110	38	410	42	590	700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	**30**
31	185	38	494	42	506	691	4	29
32	223	38	536	42	464	687	4	28
33	261	37	578	42	422	683	4	27
34	298	38	620	41	380	679	5	26
35	336	38	661	42	339	674	4	25
36	374	37	703	42	297̲	670	4	24
37	411	38	745	42	255̅	666	4	23
38	449	37	787	42	213	662	5	22
39	486	37	829	41	171	657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	**20**
41	561	37	912	41	088	649̲	4	19
42	598	37	953	42	047̲	645̅	5	18
43	635	38	995	42	005̅	640	4	17
44	673	37	9.53 037	41	0.46 963	636	4	16
45	710	37	078	42	922	632	4	15
46	747	37	120	41	880	628	5	14
47	784	37	161	41	839	623	4	13
48	821	37	202	42	798	619̲	4	12
49	858	38	244	41	756̲	615̅	5	11
50	9.50 896	37	9.53 285	42	0.46 715̅	9.97 610	4	**10**
51	933	37	327	41	673	606	4	9
52	970	37	368	41	632	602	5	8
53	9.51 007	36	409	41	591	597	4	7
54	043	37	450	42	550	593	4	6
55	080	37	492	41	508	589	5	5
56	117	37	533	41	467	584	4	4
57	154	37	574	41	426̲	580	4	3
58	191	36	615	41	385̅	576	5	2
59	227	37	656	41	344	571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	43	42
1″	0.7	0.7
2″	1.4	1.4
3″	2.2	2.1
4″	2.9	2.8
5″	3.6	3.5
6″	4.3	4.2
7″	5.0	4.9
8″	5.7	5.6
9″	6.5	6.3

1′	41	39
1″	0.7	0.7
2″	1.4	1.3
3″	2.1	2.0
4″	2.7	2.6
5″	3.4	3.3
6″	4.1	3.9
7″	4.8	4.6
8″	5.5	5.2
9″	6.2	5.9

1′	38
1″	0.6
2″	1.3
3″	1.9
4″	2.5
5″	3.2
6″	3.8
7″	4.4
8″	5.1
9″	5.7

1′	37
1″	0.6
2″	1.2
3″	1.9
4″	2.5
5″	3.1
6″	3.7
7″	4.3
8″	4.9
9″	5.6

1′	36
1″	0.6
2″	1.2
3″	1.8
4″	2.4
5″	3.0
6″	3.6
7″	4.2
8″	4.8
9″	5.4

1′	4	5
1″	0.07	0.08
2″	0.13	0.17
3″	0.20	0.25
4″	0.27	0.33
5″	0.33	0.42
6″	0.40	0.50
7″	0.47	0.58
8″	0.53	0.67
9″	0.60	0.75

71°

Logarithmic Functions

19°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	**60**
1	301	37	738	41	262	563	5	59
2	338	36	779	41	221	558	4	58
3	374	37	820	41	180	554	4	57
4	411	36	861	41	139	550	5	56
5	447	37	902	41	098	545	4	55
6	484	36	943	41	057	541	5	54
7	520	37	984	41	016	536	4	53
8	557	36	9.54 025	40	0.45 975	532	4	52
9	593	36	065	41	935	528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	4	**50**
11	666	36	147	40	853	519	4	49
12	702	36	187	41	813	515	5	48
13	738	36	228	41	772	510	4	47
14	774	37	269	40	731	506	5	46
15	811	36	309	41	691	501	4	45
16	847	36	350	40	650	497	5	44
17	883	36	390	41	610	492	4	43
18	919	36	431	40	569	488	4	42
19	955	36	471	41	529	484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	**40**
21	9.52 027	36	552	41	448	475	5	39
22	063	36	593	40	407	470	4	38
23	099	36	633	40	367	466	5	37
24	135	36	673	41	327	461	4	36
25	171	36	714	40	286	457	4	35
26	207	35	754	40	246	453	5	34
27	242	36	794	41	206	448	4	33
28	278	36	835	40	165	444	5	32
29	314	36	875	40	125	439	4	31
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	5	**30**
31	385	36	955	40	045	430	4	29
32	421	35	995	40	005	426	5	28
33	456	36	9.55 035	40	0.44 965	421	4	27
34	492	35	075	40	925	417	5	26
35	527	36	115	40	885	412	4	25
36	563	35	155	40	845	408	5	24
37	598	36	195	40	805	403	4	23
38	634	35	235	40	765	399	5	22
39	669	36	275	40	725	394	4	21
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	5	**20**
41	740	35	355	40	645	385	4	19
42	775	36	395	39	605	381	5	18
43	811	35	434	40	566	376	4	17
44	846	35	474	40	526	372	5	16
45	881	35	514	40	486	367	4	15
46	916	35	554	39	446	363	5	14
47	951	35	593	40	407	358	5	13
48	986	35	633	40	367	353	4	12
49	9.53 021	35	673	39	327	349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	**10**
51	092	34	752	39	248	340	5	9
52	126	35	791	40	209	335	4	8
53	161	35	831	39	169	331	5	7
54	196	35	870	40	130	326	4	6
55	231	35	910	39	090	322	5	5
56	266	35	949	40	051	317	5	4
57	301	35	989	39	011	312	4	3
58	336	34	9.56 028	39	0.43 972	308	5	2
59	370	35	067	40	933	303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	41	40
1″	0.7	0.7
2″	1.4	1.3
3″	2.1	2.0
4″	2.7	2.7
5″	3.4	3.3
6″	4.1	4.0
7″	4.8	4.7
8″	5.5	5.3
9″	6.2	6.0

1′	39	37
1″	0.7	0.6
2″	1.3	1.2
3″	2.0	1.9
4″	2.6	2.5
5″	3.3	3.1
6″	3.9	3.7
7″	4.6	4.3
8″	5.2	4.9
9″	5.9	5.6

1′	36	35
1″	0.6	0.6
2″	1.2	1.2
3″	1.8	1.8
4″	2.4	2.3
5″	3.0	2.9
6″	3.6	3.5
7″	4.2	4.1
8″	4.8	4.7
9″	5.4	5.3

1′	34
1″	0.6
2″	1.1
3″	1.7
4″	2.3
5″	2.8
6″	3.4
7″	4.0
8″	4.5
9″	5.1

1′	4	5
1″	0.07	0.08
2″	0.13	0.17
3″	0.20	0.25
4″	0.27	0.33
5″	0.33	0.42
6″	0.40	0.50
7″	0.47	0.58
8″	0.53	0.67
9″	0.60	0.75

70°

20°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	**60**
1	440	35	146	39	854	294	5	59
2	475	34	185	39	815	289	4	58
3	509	35	224	40	776	285	5	57
4	544	34	264	39	736	280	4	56
5	578	35	303	39	697	276	5	55
6	613	34	342	39	658	271	5	54
7	647	35	381	39	619	266	4	53
8	682	34	420	39	580	262	5	52
9	716	35	459	39	541	257	5	51
10	9.53 751	34	9.56 498	39	0.43 502	9.97 252	4	**50**
11	785	34	537	39	463	248	5	49
12	819	35	576	39	424	243	5	48
13	854	34	615	39	385	238	4	47
14	888	34	654	39	346	234	5	46
15	922	35	693	39	307	229	5	45
16	957	34	732	39	268	224	4	44
17	991	34	771	39	229	220	5	43
18	9.54 025	34	810	39	190	215	5	42
19	059	34	849	38	151	210	4	41
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	5	**40**
21	127	34	926	39	074	201	5	39
22	161	34	965	39	035	196	4	38
23	195	34	9.57 004	38	0.42 996	192	5	37
24	229	34	042	39	958	187	5	36
25	263	34	081	39	919	182	4	35
26	297	34	120	38	880	178	5	34
27	331	34	158	39	842	173	5	33
28	365	34	197	38	803	168	5	32
29	399	34	235	39	765	163	4	31
30	9.54 433	33	9.57 274	38	0.42 726	9.97 159	5	**30**
31	466	34	312	39	688	154	5	29
32	500	34	351	38	649	149	4	28
33	534	33	389	39	611	145	5	27
34	567	34	428	38	572	140	5	26
35	601	34	466	38	534	135	5	25
36	635	33	504	39	496	130	4	24
37	668	34	543	38	457	126	5	23
38	702	33	581	38	419	121	5	22
39	735	34	619	39	381	116	5	21
40	9.54 769	33	9.57 658	38	0.42 342	9.97 111	4	**20**
41	802	34	696	38	304	107	5	19
42	836	33	734	38	266	102	5	18
43	869	34	772	38	228	097	5	17
44	903	33	810	39	190	092	5	16
45	936	33	849	38	151	087	4	15
46	969	34	887	38	113	083	5	14
47	9.55 003	33	925	38	075	078	5	13
48	036	33	963	38	037	073	5	12
49	069	33	9.58 001	38	0.41 999	068	5	11
50	9.55 102	34	9.58 039	38	0.41 961	9.97 063	4	**10**
51	136	33	077	38	923	059	5	9
52	169	33	115	38	885	054	5	8
53	202	33	153	38	847	049	5	7
54	235	33	191	38	809	044	5	6
55	268	33	229	38	771	039	4	5
56	301	33	267	37	733	035	5	4
57	334	33	304	38	696	030	5	3
58	367	33	342	38	658	025	5	2
59	400	33	380	38	620	020	5	1
60	9.55 433		9.58 418		0.41 582	9.97 015		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	40	39
1″	0.7	0.7
2″	1.3	1.3
3″	2.0	2.0
4″	2.7	2.6
5″	3.3	3.3
6″	4.0	3.9
7″	4.7	4.6
8″	5.3	5.2
9″	6.0	5.9

1′	38	37
1″	0.6	0.6
2″	1.3	1.2
3″	1.9	1.9
4″	2.5	2.5
5″	3.2	3.1
6″	3.8	3.7
7″	4.4	4.3
8″	5.1	4.9
9″	5.7	5.6

1′	35
1″	0.6
2″	1.2
3″	1.8
4″	2.3
5″	2.9
6″	3.5
7″	4.1
8″	4.7
9″	5.3

1′	34
1″	0.6
2″	1.1
3″	1.7
4″	2.3
5″	2.8
6″	3.4
7″	4.0
8″	4.5
9″	5.1

1′	33
1″	0.6
2″	1.1
3″	1.7
4″	2.2
5″	2.8
6″	3.3
7″	3.9
8″	4.4
9″	5.0

1′	4	5
1″	0.07	0.08
2″	0.13	0.17
3″	0.20	0.25
4″	0.27	0.33
5″	0.33	0.42
6″	0.40	0.50
7″	0.47	0.58
8″	0.53	0.67
9″	0.60	0.75

69°

Logarithmic Functions

21°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	**60**
1	466	33	455	38	$54\bar{5}$	010	5	59
2	499	33	493	38	507	005	4	58
3	532	32	531	38	469	001	5	57
4	564	33	569	37	431	9.96 996	5	56
5	597	33	606	38	394	991	5	55
6	630	33	644	37	356	986	5	54
7	663	32	681	38	319	981	5	53
8	695	33	719	38	281	976	5	52
9	728	33	757	37	243	971	5	51
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	4	**50**
11	793	33	832	37	168	962	5	49
12	826	32	869	38	131	957	5	48
13	858	33	907	37	093	952	5	47
14	891	32	944	37	056	947	5	46
15	923	33	981	38	019	942	5	45
16	956	32	9.59 019	37	0.40 981	937	5	44
17	988	33	056	38	944	932	5	43
18	9.56 021	32	094	37	906	927	5	42
19	053	32	131	37	869	922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	**40**
21	118	32	205	38	$79\bar{5}$	912	5	39
22	150	32	243	37	757	907	4	38
23	182	33	280	37	720	903	5	37
24	$21\bar{5}$	32	317	37	683	898	5	36
25	247	32	354	37	646	893	5	35
26	279	32	391	38	609	888	5	34
27	311	32	429	37	571	883	5	33
28	343	32	466	37	534	878	5	32
29	375	33	503	37	497	873	5	31
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	**30**
31	440	32	577	37	423	863	5	29
32	472	32	614	37	386	858	5	28
33	504	32	651	37	349	853	5	27
34	536	32	$68\underline{8}$	37	312	848	5	26
35	568	31	$72\bar{5}$	37	275	843	5	25
36	599	32	762	37	238	838	5	24
37	631	32	799	36	$20\underline{1}$	833	5	23
38	663	32	835	37	165	828	5	22
39	695	32	872	37	128	823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	**20**
41	759	31	946	37	054	813	5	19
42	790	32	983	36	017	808	5	18
43	822	32	9.60 019	37	0.39 981	803	5	17
44	854	32	056	37	944	798	5	16
45	886	31	093	37	907	793	5	15
46	917	32	130	36	870	788	5	14
47	949	31	166	37	834	783	5	13
48	980	32	203	37	797	778	6	12
49	9.57 012	32	240	36	760	772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	**10**
51	075	32	313	36	687	762	5	9
52	107	31	349	37	651	757	5	8
53	138	31	386	36	614	752	5	7
54	169	32	422	37	578	747	5	6
55	201	31	459	36	541	742	5	5
56	232	32	495	37	$50\bar{5}$	737	5	4
57	$26\underline{4}$	31	532	36	468	732	5	3
58	295	31	$56\underline{8}$	37	432	727	5	2
59	326	32	605	36	395	722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	38	37
1″	0.6	0.6
2″	1.3	1.2
3″	1.9	1.9
4″	2.5	2.5
5″	3.2	3.1
6″	3.8	3.7
7″	4.4	4.3
8″	5.1	4.9
9″	5.7	5.6

1′	36	33
1″	0.6	0.6
2″	1.2	1.1
3″	1.8	1.7
4″	2.4	2.2
5″	3.0	2.8
6″	3.6	3.3
7″	4.2	3.9
8″	4.8	4.4
9″	5.4	5.0

1′	32
1″	0.5
2″	1.1
3″	1.6
4″	2.1
5″	2.7
6″	3.2
7″	3.7
8″	4.3
9″	4.8

1′	31
1″	0.5
2″	1.0
3″	1.6
4″	2.1
5″	2.6
6″	3.1
7″	3.6
8″	4.1
9″	4.7

1′	4
1″	0.07
2″	0.13
3″	0.20
4″	0.27
5″	0.33
6″	0.40
7″	0.47
8″	0.53
9″	0.60

1′	5	6
1″	0.08	0.10
2″	0.17	0.20
3″	0.25	0.30
4″	0.33	0.40
5″	0.42	0.50
6″	0.50	0.60
7″	0.58	0.70
8″	0.67	0.80
9″	0.75	0.90

68°

Logarithmic Functions

22°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	**60**
1	389	31	677	37	323	711	5	59
2	420	31	714	36	286	706	5	58
3	451	31	750	36	250	701	5	57
4	482	32	786	37	214	696	5	56
5	514	31	823	36	177	691	5	55
6	54$\bar{5}$	31	859	36	14$\underline{1}$	686	5	54
7	576	31	895	36	10$\bar{5}$	681	5	53
8	607	31	931	36	069	676	6	52
9	638	31	967	37	033	670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	**50**
11	700	31	040	36	960	660	5	49
12	731	31	076	36	924	655	5	48
13	762	31	112	36	888	65$\underline{0}$	5	47
14	793	31	148	36	852	64$\bar{5}$	5	46
15	824	31	184	36	816	640	6	45
16	85$\bar{5}$	30	220	36	780	634	5	44
17	885	31	256	36	744	629	5	43
18	916	31	292	36	708	624	5	42
19	947	31	328	36	672	619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	**40**
21	9.58 008	31	400	36	600	608	5	39
22	039	31	436	36	564	603	5	38
23	070	31	472	36	528	598	5	37
24	101	30	508	36	492	593	5	36
25	131	31	544	35	456	588	6	35
26	162	30	579	36	42$\underline{1}$	582	5	34
27	192	31	615	36	38$\bar{5}$	577	5	33
28	223	30	651	36	349	572	5	32
29	253	31	687	35	313	567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	**30**
31	31$\underline{4}$	31	758	36	242	556	5	29
32	34$\bar{5}$	30	794	36	206	551	5	28
33	375	31	830	35	17$\underline{0}$	546	5	27
34	406	30	865	36	13$\bar{5}$	541	6	26
35	436	31	901	35	099	535	5	25
36	467	30	936	36	064	53$\underline{0}$	5	24
37	497	30	972	36	028	52$\bar{5}$	5	23
38	527	30	9.62 008	35	0.37 992	520	6	22
39	557	31	043	36	957	514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	**20**
41	618	30	114	36	886	504	6	19
42	648	30	150	35	85$\underline{0}$	498	5	18
43	678	31	185	36	81$\bar{5}$	493	5	17
44	709	30	221	35	779	488	5	16
45	739	30	256	36	744	483	6	15
46	769	30	292	35	708	477	5	14
47	799	30	327	35	673	472	5	13
48	829	30	362	36	638	467	6	12
49	859	30	398	35	602	461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	**10**
51	919	30	468	36	532	451	6	9
52	949	30	504	35	496	445	5	8
53	979	30	539	35	461	44$\underline{0}$	5	7
54	9.59 009	30	574	35	426	43$\bar{5}$	6	6
55	039	30	609	36	391	429	5	5
56	069	29	64$\bar{5}$	35	355	424	5	4
57	098	30	68$\underline{0}$	35	320	419	6	3
58	128	30	71$\bar{5}$	35	285	413	5	2
59	158	30	750	35	25$\underline{0}$	408	5	1
60	9.59 188		9.62 785		0.37 21$\bar{5}$	9.96 403		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.		
1′	37	36
1″	0.6	0.6
2″	1.2	1.2
3″	1.9	1.8
4″	2.5	2.4
5″	3.1	3.0
6″	3.7	3.6
7″	4.3	4.2
8″	4.9	4.8
9″	5.6	5.4
1′	35	32
1″	0.6	0.5
2″	1.2	1.1
3″	1.8	1.6
4″	2.3	2.1
5″	2.9	2.7
6″	3.5	3.2
7″	4.1	3.7
8″	4.7	4.3
9″	5.3	4.8
1′	31	
1″	0.5	
2″	1.0	
3″	1.6	
4″	2.1	
5″	2.6	
6″	3.1	
7″	3.6	
8″	4.1	
9″	4.7	
1′	30	
1″	0.5	
2″	1.0	
3″	1.5	
4″	2.0	
5″	2.5	
6″	3.0	
7″	3.5	
8″	4.0	
9″	4.5	
1′	29	
1″	0.5	
2″	1.0	
3″	1.5	
4″	1.9	
5″	2.4	
6″	2.9	
7″	3.4	
8″	3.9	
9″	4.4	
1′	5	6
1″	0.08	0.10
2″	0.17	0.20
3″	0.25	0.30
4″	0.33	0.40
5″	0.42	0.50
6″	0.50	0.60
7″	0.58	0.70
8″	0.67	0.80
9″	0.75	0.90

67°

Logarithmic Functions

23°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.59 188	30	9.62 785	35	0.37 215	9.96 403	6	**60**
1	218	29	820	35	180	397	5	59
2	247	30	855	35	145	392	5	58
3	277	30	890	36	110	387	6	57
4	307	29	926	35	074	381	5	56
5	336	30	961	35	039	376	6	55
6	366	30	996	35	004	370	5	54
7	396	29	9.63 031	35	0.36 969	365	5	53
8	425	30	066	35	934	360	6	52
9	455	29	101	34	899	354	5	51
10	9.59 484	30	9.63 135	35	0.36 865	9.96 349	6	**50**
11	514	29	170	35	830	343	5	49
12	543	30	205	35	795	338	5	48
13	573	29	240	35	760	333	6	47
14	602	30	275	35	725	327	5	46
15	632	29	310	35	690	322	6	45
16	661	29	345	34	655	316	5	44
17	690	30	379	35	621	311	6	43
18	720	29	414	35	586	305	5	42
19	749	29	449	35	551	300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	**40**
21	808	29	519	34	481	289	5	39
22	837	29	553	35	447	284	6	38
23	866	29	588	35	412	278	5	37
24	895	29	623	34	377	273	6	36
25	924	30	657	35	343	267	5	35
26	954	29	692	34	308	262	6	34
27	983	29	726	35	274	256	5	33
28	9.60 012	29	761	35	239	251	6	32
29	041	29	796	34	204	245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	**30**
31	099	29	865	34	135	234	5	29
32	128	29	899	35	101	229	6	28
33	157	29	934	34	066	223	5	27
34	186	29	968	35	032	218	6	26
35	215	29	9.64 003	34	0.35 997	212	5	25
36	244	29	037	35	963	207	6	24
37	273	29	072	34	928	201	5	23
38	302	29	106	34	894	196	6	22
39	331	28	140	35	860	190	5	21
40	9.60 359	29	9.64 175	34	0.35 825	9.96 185	6	**20**
41	388	29	209	34	791	179	5	19
42	417	29	243	35	757	174	6	18
43	446	28	278	34	722	168	6	17
44	474	29	312	34	688	162	5	16
45	503	29	346	35	654	157	6	15
46	532	29	381	34	619	151	5	14
47	561	28	415	34	585	146	6	13
48	589	29	449	34	551	140	5	12
49	618	28	483	34	517	135	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	**10**
51	675	29	552	34	448	123	5	9
52	704	28	586	34	414	118	6	8
53	732	29	620	34	380	112	5	7
54	761	28	654	34	346	107	6	6
55	789	29	688	34	312	101	6	5
56	818	28	722	34	278	095	5	4
57	846	29	756	34	244	090	6	3
58	875	28	790	34	210	084	5	2
59	903	28	824	34	176	079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	36	35
1″	0.6	0.6
2″	1.2	1.2
3″	1.8	1.8
4″	2.4	2.3
5″	3.0	2.9
6″	3.6	3.5
7″	4.2	4.1
8″	4.8	4.7
9″	5.4	5.3

1′	34	30
1″	0.6	0.5
2″	1.1	1.0
3″	1.7	1.5
4″	2.3	2.0
5″	2.8	2.5
6″	3.4	3.0
7″	4.0	3.5
8″	4.5	4.0
9″	5.1	4.5

1′	29
1″	0.5
2″	1.0
3″	1.5
4″	1.9
5″	2.4
6″	2.9
7″	3.4
8″	3.9
9″	4.4

1′	28
1″	0.5
2″	0.9
3″	1.4
4″	1.9
5″	2.3
6″	2.8
7″	3.3
8″	3.7
9″	4.2

1′	5	6
1″	0.08	0.10
2″	0.17	0.20
3″	0.25	0.30
4″	0.33	0.40
5″	0.42	0.50
6″	0.50	0.60
7″	0.58	0.70
8″	0.67	0.80
9″	0.75	0.90

66°

24°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	60
1	960	28	892	34	108	067	5	59
2	988	28	926	34	074	062	6	58
3	9.61 01$\underline{6}$	29	960	34	040	056	6	57
4	04$\bar{5}$	28	994	34	006	05$\underline{0}$	5	56
5	073	28	9.65 028	34	0.34 972	04$\bar{5}$	6	55
6	101	28	062	34	938	039	5	54
7	129	29	096	34	904	034	6	53
8	158	28	130	34	870	028	6	52
9	186	28	164	33	836	022	5	51
10	9.61 214	28	9.65 197	34	0.34 803	9.96 017	6	50
11	242	28	231	34	76$\underline{9}$	011	6	49
12	270	28	265	34	73$\bar{5}$	005	5	48
13	298	28	299	34	701	000	6	47
14	326	28	333	33	667	9.95 994	6	46
15	354	28	366	34	634	988	6	45
16	382	29	400	34	600	982	5	44
17	411	27	434	33	566	977	6	43
18	438	28	467	34	533	971	6	42
19	466	28	50$\underline{1}$	34	499	965	5	41
20	9.61 494	28	9.65 53$\bar{5}$	33	0.34 465	9.95 960	6	40
21	522	28	568	34	432	954	6	39
22	550	28	602	34	398	948	6	38
23	578	28	636	33	364	942	5	37
24	606	28	669	34	331	937	6	36
25	634	28	703	33	297	931	6	35
26	662	27	736	34	264	925	5	34
27	689	28	770	33	230	920	6	33
28	71$\underline{7}$	28	803	34	197	914	6	32
29	74$\bar{5}$	28	837	33	163	908	6	31
30	9.61 773	27	9.65 870	34	0.34 130	9.95 902	5	30
31	800	28	904	33	096	897	6	29
32	828	28	937	34	063	89$\underline{1}$	6	28
33	856	27	971	33	029	88$\bar{5}$	6	27
34	883	28	9.66 004	34	0.33 996	879	6	26
35	911	28	038	33	962	873	5	25
36	939	27	071	33	929	868	6	24
37	966	28	104	34	896	862	6	23
38	994	27	138	33	862	856	6	22
39	9.62 021	28	171	33	829	850	6	21
40	9.62 049	27	9.66 204	34	0.33 796	9.95 844	5	20
41	076	28	238	33	762	839	6	19
42	104	27	271	33	729	833	6	18
43	131	28	304	33	696	827	6	17
44	159	27	337	34	663	821	6	16
45	186	28	371	33	629	815	5	15
46	214	27	404	33	596	810	6	14
47	241	27	437	33	563	804	6	13
48	268	28	470	33	530	798	6	12
49	296	27	503	34	497	792	6	11
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	10
51	350	27	570	33	430	78$\underline{0}$	5	9
52	37$\underline{7}$	28	603	33	397	77$\bar{5}$	6	8
53	40$\bar{5}$	27	636	33	364	769	6	7
54	432	27	669	33	331	763	6	6
55	459	27	702	33	298	757	6	5
56	486	27	735	33	26$\bar{5}$	751	6	4
57	513	28	768	33	232	745	6	3
58	541	27	801	33	199	739	6	2
59	56$\underline{8}$	27	834	33	166	733	5	1
60	9.62 59$\bar{5}$		9.66 867		0.33 133	9.95 728		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	34	33
1″	0.6	0.6
2″	1.1	1.1
3″	1.7	1.7
4″	2.3	2.2
5″	2.8	2.8
6″	3.4	3.3
7″	4.0	3.9
8″	4.5	4.4
9″	5.1	5.0

1′	29
1″	0.5
2″	1.0
3″	1.5
4″	1.9
5″	2.4
6″	2.9
7″	3.4
8″	3.9
9″	4.4

1′	28
1″	0.5
2″	0.9
3″	1.4
4″	1.9
5″	2.3
6″	2.8
7″	3.3
8″	3.7
9″	4.2

1′	27
1″	0.5
2″	0.9
3″	1.4
4″	1.8
5″	2.3
6″	2.7
7″	3.2
8″	3.6
9″	4.1

1′	5	6
1″	0.08	0.10
2″	0.17	0.20
3″	0.25	0.30
4″	0.33	0.40
5″	0.42	0.50
6″	0.50	0.60
7″	0.58	0.70
8″	0.67	0.80
9″	0.75	0.90

65°

25°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.62 595	27	9.66 867	33	0.33 133	9.95 728	6	60
1	622	27	900	33	100	722	6	59
2	649	27	933	33	067	716	6	58
3	676	27	966	33	034	710	6	57
4	703	27	999	33	001	704	6	56
5	730	27	9.67 032	33	0.32 968	698	6	55
6	757	27	065	33	935	692	6	54
7	784	27	098	33	902	686	6	53
8	811	27	131	32	869	680	6	52
9	838	27	163	33	837	674	6	51
10	9.62 865	27	9.67 196	33	0.32 804	9.95 668	5	50
11	892	26	229	33	771	663	6	49
12	918	27	262	33	738	657	6	48
13	945	27	295	32	705	651	6	47
14	972	27	327	33	673	645	6	46
15	999	27	360	33	640	639	6	45
16	9.63 026	26	393	33	607	633	6	44
17	052	27	426	32	574	627	6	43
18	079	27	458	33	542	621	6	42
19	106	27	491	33	509	615	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	40
21	159	27	556	33	444	603	6	39
22	186	27	589	33	411	597	6	38
23	213	26	622	32	378	591	6	37
24	239	27	654	33	346	585	6	36
25	266	26	687	32	313	579	6	35
26	292	27	719	33	281	573	6	34
27	319	26	752	33	248	567	6	33
28	345	27	785	32	215	561	6	32
29	372	26	817	33	183	555	6	31
30	9.63 398	27	9.67 850	32	0.32 150	9.95 549	6	30
31	425	26	882	33	118	543	6	29
32	451	27	915	32	085	537	6	28
33	478	26	947	33	053	531	6	27
34	504	27	980	32	020	525	6	26
35	531	26	9.68 012	32	0.31 988	519	6	25
36	557	26	044	33	956	513	6	24
37	583	27	077	32	923	507	7	23
38	610	26	109	33	891	500	6	22
39	636	26	142	32	858	494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	20
41	689	26	206	33	794	482	6	19
42	715	26	239	32	761	476	6	18
43	741	26	271	32	729	470	6	17
44	767	27	303	33	697	464	6	16
45	794	26	336	32	664	458	6	15
46	820	26	368	32	632	452	6	14
47	846	26	400	32	600	446	6	13
48	872	26	432	33	568	440	6	12
49	898	26	465	32	535	434	7	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	6	10
51	950	26	529	32	471	421	6	9
52	976	26	561	32	439	415	6	8
53	9.64 002	26	593	33	407	409	6	7
54	028	26	626	32	374	403	6	6
55	054	26	658	32	342	397	6	5
56	080	26	690	32	310	391	7	4
57	106	26	722	32	278	384	6	3
58	132	26	754	32	246	378	6	2
59	158	26	786	32	214	372	6	1
60	9.64 184		9.68 818		0.31 182	9.95 366		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	33	32
1″	0.6	0.5
2″	1.1	1.1
3″	1.7	1.6
4″	2.2	2.1
5″	2.8	2.7
6″	3.3	3.2
7″	3.9	3.7
8″	4.4	4.3
9″	5.0	4.8

1′	27
1″	0.5
2″	0.9
3″	1.4
4″	1.8
5″	2.3
6″	2.7
7″	3.2
8″	3.6
9″	4.1

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	5
1″	0.08
2″	0.17
3″	0.25
4″	0.33
5″	0.42
6″	0.50
7″	0.58
8″	0.67
9″	0.75

1′	6	7
1″	0.10	0.12
2″	0.20	0.23
3″	0.30	0.35
4″	0.40	0.47
5″	0.50	0.58
6″	0.60	0.70
7″	0.70	0.82
8″	0.80	0.93
9″	0.90	1.05

64°

Logarithmic Functions

26°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	**60**
1	210	26	850	32	150	360	6	59
2	236	26	882	32	118	354	6	58
3	262	26	914	32	086	348	7	57
4	288	25	946	32	054	341	6	56
5	313	26	978	32	022	335	6	55
6	339	26	9.69 010	32	0.30 990	329	6	54
7	365	26	042	32	958	323	6	53
8	391	26	074	32	926	317	7	52
9	417	25	106	32	894	310	6	51
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	**50**
11	468	26	170	32	830	298	6	49
12	494	25	202	32	798	292	6	48
13	519̲	26	234	32	766	286	7	47
14	545̄	26	266	32	734	279	6	46
15	571	25	298	31	702	273	6	45
16	596	26	329	32	671	267	6	44
17	622	25	361	32	639	261	7	43
18	647	26	393̲	32	607	254	6	42
19	673	25	425̄	32	575	248	6	41
20	9.64 698	26	9.69 457	31	0.30 543	9.95 242	6	**40**
21	724	25	488	32	512	236	7	39
22	749	26	520	32	480	229	6	38
23	775̄	25	552	32	448	223	6	37
24	800	26	584	31	416̲	217	6	36
25	826	25	615	32	385̄	211	7	35
26	851	26	647	32	353	204	6	34
27	877	25	679	31	321	198	6	33
28	902	25	710	32	290	192	7	32
29	927	26	742	32	258	185	6	31
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	**30**
31	978	25	805	32	195̄	173	6	29
32	9.65 003	26	837	31	163	167	7	28
33	029	25	868	32	132	160	6	27
34	054	25	900	32	100	154	6	26
35	079	25	932	31	068	148	7	25
36	104	26	963̲	32	037	141̲	6	24
37	130̲	25	995̄	31	005	135̄	6	23
38	155̄	25	9.70 026	32	0.29 974	129	7	22
39	180	25	058	31	942	122	6	21
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	**20**
41	230	25	121	31	879	110	7	19
42	255	26	152	32	848	103	6	18
43	281	25	184	31	816̲	097	7	17
44	306	25	215	32	785̄	090	6	16
45	331	25	247	31	753	084	6	15
46	356	25	278	31	722	078	7	14
47	381	25	309	32	691	071̲	6	13
48	406	25	341	31	659	065̄	6	12
49	431	25	372	32	628	059	7	11
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	**10**
51	481	25	435̄	31	565	046	7	9
52	506	25	466	32	534	039	6	8
53	531	25	498	31	502	033	6	7
54	556	24	529	31	471	027	7	6
55	580	25	560	32	440	020	6	5
56	605	25	592	31	408	014	7	4
57	630	25	623	31	377	007	6	3
58	655	25	654	31	346̲	001̲	6	2
59	680	25	685	32	315̄	9.94 995̄	7	1
60	9.65 705̄		9.70 717		0.29 283	9.94 988		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	32	31
1″	0.5	0.5
2″	1.1	1.0
3″	1.6	1.6
4″	2.1	2.1
5″	2.7	2.6
6″	3.2	3.1
7″	3.7	3.6
8″	4.3	4.1
9″	4.8	4.7

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	25
1″	0.4
2″	0.8
3″	1.3
4″	1.7
5″	2.1
6″	2.5
7″	2.9
8″	3.3
9″	3.8

1′	24
1″	0.4
2″	0.8
3″	1.2
4″	1.6
5″	2.0
6″	2.4
7″	2.8
8″	3.2
9″	3.6

1′	6	7
1″	0.10	0.12
2″	0.20	0.23
3″	0.30	0.35
4″	0.40	0.47
5″	0.50	0.58
6″	0.60	0.70
7″	0.70	0.82
8″	0.80	0.93
9″	0.90	1.05

63°

Logarithmic Functions

27°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.65 705	24	9.70 717	31	0.29 283	9.94 988	6	**60**
1	729	25	748	31	252	982	7	59
2	754	25	779	31	221	975	6	58
3	779	25	810	31	190	969	7	57
4	804	24	841	32	159	962	6	56
5	828	25	873	31	127	956	7	55
6	853	25	904	31	096	949	6	54
7	878	24	935	31	065	943	7	53
8	902	25	966	31	034	936	6	52
9	927	25	997	31	003	930	7	51
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	6	**50**
11	976	25	059	31	941	917	6	49
12	9.66 001	24	090	31	910	911	7	48
13	025	25	121	32	879	904	6	47
14	050	25	153	31	847	898	7	46
15	075	24	184	31	816	891	6	45
16	099	25	215	31	785	885	7	44
17	124	24	246	31	754	878	7	43
18	148	25	277	31	723	871	6	42
19	173	24	308	31	692	865	7	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	6	**40**
21	221	25	370	31	630	852	7	39
22	246	24	401	30	599	845	6	38
23	270	25	431	31	569	839	7	37
24	295	24	462	31	538	832	6	36
25	319	24	493	31	507	826	7	35
26	343	25	524	31	476	819	6	34
27	368	24	555	31	445	813	7	33
28	392	24	586	31	414	806	7	32
29	416	25	617	31	383	799	6	31
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	**30**
31	465	24	679	30	321	786	6	29
32	489	24	709	31	291	780	7	28
33	513	24	740	31	260	773	6	27
34	537	25	771	31	229	767	7	26
35	562	24	802	31	198	760	7	25
36	586	24	833	30	167	753	6	24
37	610	24	863	31	137	747	7	23
38	634	24	894	31	106	740	6	22
39	658	24	925	30	075	734	7	21
40	9.66 682	24	9.71 955	31	0.28 045	9.94 727	7	**20**
41	706	25	986	31	014	720	6	19
42	731	24	9.72 017	31	0.27 983	714	7	18
43	755	24	048	30	952	707	7	17
44	779	24	078	31	922	700	6	16
45	803	24	109	31	891	694	7	15
46	827	24	140	30	860	687	7	14
47	851	24	170	31	830	680	6	13
48	875	24	201	30	799	674	7	12
49	899	23	231	31	769	667	7	11
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	6	**10**
51	946	24	293	30	707	654	7	9
52	970	24	323	31	677	647	7	8
53	994	24	354	30	646	640	6	7
54	9.67 018	24	384	31	616	634	7	6
55	042	24	415	30	585	627	7	5
56	066	24	445	31	555	620	6	4
57	090	23	476	30	524	614	7	3
58	113	24	506	31	494	607	7	2
59	137	24	537	30	463	600	7	1
60	9.67 161		9.72 567		0.27 433	9.94 593		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	32	31
1″	0.5	0.5
2″	1.1	1.0
3″	1.6	1.6
4″	2.1	2.1
5″	2.7	2.6
6″	3.2	3.1
7″	3.7	3.6
8″	4.3	4.1
9″	4.8	4.7

1′	30	25
1″	0.5	0.4
2″	1.0	0.8
3″	1.5	1.3
4″	2.0	1.7
5″	2.5	2.1
6″	3.0	2.5
7″	3.5	2.9
8″	4.0	3.3
9″	4.5	3.8

1′	24
1″	0.4
2″	0.8
3″	1.2
4″	1.6
5″	2.0
6″	2.4
7″	2.8
8″	3.2
9″	3.6

1′	23
1″	0.4
2″	0.8
3″	1.2
4″	1.5
5″	1.9
6″	2.3
7″	2.7
8″	3.1
9″	3.5

1′	6	7
1″	0.10	0.12
2″	0.20	0.23
3″	0.30	0.35
4″	0.40	0.47
5″	0.50	0.58
6″	0.60	0.70
7″	0.70	0.82
8″	0.80	0.93
9″	0.90	1.05

62°

Logarithmic Functions

28°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60
1	185	23	598	30	402	587	7	59
2	208	24	628	31	372	580	7	58
3	232	24	659	30	341	573	6	57
4	256	24	689	31	311	567	7	56
5	280	23	720	30	280	560	7	55
6	303	24	750	30	250	553	7	54
7	327	23	780	31	220	546	6	53
8	350	24	811	30	189	540	7	52
9	374	24	841	31	159	533	7	51
10	9.67 398	23	9.72 872	30	0.27 128	9.94 526	7	50
11	421	24	902	30	098	519	6	49
12	445	23	932	31	068	513	7	48
13	468	24	963	30	037	506	7	47
14	492	23	993	30	007	499	7	46
15	515	24	9.73 023	31	0.26 977	492	7	45
16	539	23	054	30	946	485	6	44
17	562	24	084	30	916	479	7	43
18	586	23	114	30	886	472	7	42
19	609	24	144	31	856	465	7	41
20	9.67 633	23	9.73 175	30	0.26 825	9.94 458	7	40
21	656	24	205	30	795	451	6	39
22	680	23	235	30	765	445	7	38
23	703	23	265	30	735	438	7	37
24	726	24	295	31	705	431	7	36
25	750	23	326	30	674	424	7	35
26	773	23	356	30	644	417	7	34
27	796	24	386	30	614	410	6	33
28	820	23	416	30	584	404	7	32
29	843	23	446	30	554	397	7	31
30	9.67 866	24	9.73 476	31	0.26 524	9.94 390	7	30
31	890	23	507	30	493	383	7	29
32	913	23	537	30	463	376	7	28
33	936	23	567	30	433	369	7	27
34	959	23	597	30	403	362	7	26
35	982	24	627	30	373	355	6	25
36	9.68 006	23	657	30	343	349	7	24
37	029	23	687	30	313	342	7	23
38	052	23	717	30	283	335	7	22
39	075	23	747	30	253	328	7	21
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
41	121	23	807	30	193	314	7	19
42	144	23	837	30	163	307	7	18
43	167	23	867	30	133	300	7	17
44	190	23	897	30	103	293	7	16
45	213	24	927	30	073	286	7	15
46	237	23	957	30	043	279	6	14
47	260	23	987	30	013	273	7	13
48	283	22	9.74 017	30	0.25 983	266	7	12
49	305	23	047	30	953	259	7	11
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
51	351	23	107	30	893	245	7	9
52	374	23	137	29	863	238	7	8
53	397	23	166	30	834	231	7	7
54	420	23	196	30	804	224	7	6
55	443	23	226	30	774	217	7	5
56	466	23	256	30	744	210	7	4
57	489	23	286	30	714	203	7	3
58	512	22	316	29	684	196	7	2
59	534	23	345	30	655	189	7	1
60	9.68 557		9.74 375		0.25 625	9.94 182		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	31	30
1″	0.5	0.5
2″	1.0	1.0
3″	1.6	1.5
4″	2.1	2.0
5″	2.6	2.5
6″	3.1	3.0
7″	3.6	3.5
8″	4.1	4.0
9″	4.7	4.5

1′	29	24
1″	0.5	0.4
2″	1.0	0.8
3″	1.5	1.2
4″	1.9	1.6
5″	2.4	2.0
6″	2.9	2.4
7″	3.4	2.8
8″	3.9	3.2
9″	4.4	3.6

1′	23
1″	0.4
2″	0.8
3″	1.2
4″	1.5
5″	1.9
6″	2.3
7″	2.7
8″	3.1
9″	3.5

1′	22
1″	0.4
2″	0.7
3″	1.1
4″	1.5
5″	1.8
6″	2.2
7″	2.6
8″	2.9
9″	3.3

1′	6	7
1″	0.10	0.12
2″	0.20	0.23
3″	0.30	0.35
4″	0.40	0.47
5″	0.50	0.58
6″	0.60	0.70
7″	0.70	0.82
8″	0.80	0.93
9″	0.90	1.05

61°

Logarithmic Functions

29°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	60
1	580	23	405	30	595	175	7	59
2	603	22	435	30	565	168	7	58
3	625	23	465	29	535	161	7	57
4	648	23	494	30	506	154	7	56
5	671	23	524	30	476	147	7	55
6	694	22	554	29	446	140	7	54
7	716	23	583	30	417	133	7	53
8	739	23	613	30	387	126	7	52
9	762	22	643	30	357	119	7	51
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	**50**
11	807	22	702	30	298	105	7	49
12	829	23	732	30	268	098	8	48
13	852	23	762	29	238	090	7	47
14	875	22	791	30	209	083	7	46
15	897	23	821	30	179	076	7	45
16	920	22	851	29	149	069	7	44
17	942	23	880	30	120	062	7	43
18	965	22	910	29	090	055	7	42
19	987	23	939	30	061	048	7	41
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	**40**
21	032	23	998	30	002	034	7	39
22	055	22	9.75 028	30	0.24 972	027	7	38
23	077	23	058	29	942	020	8	37
24	100	22	087	30	913	012	7	36
25	122	22	117	29	883	005	7	35
26	144	23	146	30	854	9.93 998	7	34
27	167	22	176	29	824	991	7	33
28	189	23	205	30	795	984	7	32
29	212	22	235	29	765	977	7	31
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	**30**
31	256	23	294	29	706	963	8	29
32	279	22	323	30	677	955	7	28
33	301	22	353	29	647	948	7	27
34	323	22	382	29	618	941	7	26
35	345	23	411	30	589	934	7	25
36	368	22	441	29	559	927	7	24
37	390	22	470	30	530	920	8	23
38	412	22	500	29	500	912	7	22
39	434	22	529	29	471	905	7	21
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	**20**
41	479	22	588	29	412	891	7	19
42	501	22	617	30	383	884	8	18
43	523	22	647	29	353	876	7	17
44	545	22	676	29	324	869	7	16
45	567	22	705	30	295	862	7	15
46	589	22	735	29	265	855	8	14
47	611	22	764	29	236	847	7	13
48	633	22	793	29	207	840	7	12
49	655	22	822	30	178	833	7	11
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	**10**
51	699	22	881	29	119	819	8	9
52	721	22	910	29	090	811	7	8
53	743	22	939	30	061	804	7	7
54	765	22	969	29	031	797	8	6
55	787	22	998	29	002	789	7	5
56	809	22	9.76 027	29	0.23 973	782	7	4
57	831	22	056	30	944	775	7	3
58	853	22	086	29	914	768	8	2
59	875	22	115	29	885	760	7	1
60	9.69 897		9.76 144		0.23 856	9.93 753		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	30
1″	0.5
2″	1.0
3″	1.5
4″	2.0
5″	2.5
6″	3.0
7″	3.5
8″	4.0
9″	4.5

1′	29
1″	0.5
2″	1.0
3″	1.5
4″	1.9
5″	2.4
6″	2.9
7″	3.4
8″	3.9
9″	4.4

1′	23
1″	0.4
2″	0.8
3″	1.2
4″	1.5
5″	1.9
6″	2.3
7″	2.7
8″	3.1
9″	3.5

1′	22
1″	0.4
2″	0.7
3″	1.1
4″	1.5
5″	1.8
6″	2.2
7″	2.6
8″	2.9
9″	3.3

1′	7	8
1″	0.12	0.13
2″	0.23	0.27
3″	0.35	0.40
4″	0.47	0.53
5″	0.58	0.67
6″	0.70	0.80
7″	0.82	0.93
8″	0.93	1.07
9″	1.05	1.20

60°

Logarithmic Functions

30°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	60
1	919	22	173	29	827	746	8	59
2	941	22	202	29	798	738	7	58
3	963	21	231	30	769	731	7	57
4	984	22	261	29	739	724	7	56
5	9.70 006	22	290	29	710	717	8	55
6	028	22	319	29	681	709	7	54
7	050	22	348	29	652	702	7	53
8	072	21	377	29	623	695	8	52
9	093	22	406	29	594	687	7	51
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	50
11	137	22	464	29	536	673	8	49
12	159	21	493	29	507	665	7	48
13	180	22	522	29	478	658	8	47
14	202	22	551	29	449	650	7	46
15	224	21	580	29	420	643	7	45
16	245	22	609	30	391	636	8	44
17	267	21	639	29	361	628	7	43
18	288	22	668	29	332	621	7	42
19	310	22	697	28	303	614	8	41
20	9.70 332	21	9.76 725	29	0.23 275	9.93 606	7	40
21	353	22	754	29	246	599	8	39
22	375	21	783	29	217	591	7	38
23	396	22	812	29	188	584	7	37
24	418	21	841	29	159	577	8	36
25	439	22	870	29	130	569	7	35
26	461	21	899	29	101	562	8	34
27	482	22	928	29	072	554	7	33
28	504	21	957	29	043	547	8	32
29	525	22	986	29	014	539	7	31
30	9.70 547	21	9.77 015	29	0.22 985	9.93 532	7	30
31	568	22	044	29	956	525	8	29
32	590	21	073	28	927	517	7	28
33	611	22	101	29	899	510	8	27
34	633	21	130	29	870	502	7	26
35	654	21	159	29	841	495	8	25
36	675	22	188	29	812	487	7	24
37	697	21	217	29	783	480	8	23
38	718	21	246	28	754	472	7	22
39	739	22	274	29	726	465	8	21
40	9 70 761	21	9.77 303	29	0.22 697	9.93 457	7	20
41	782	21	332	29	668	450	8	19
42	803	21	361	29	639	442	7	18
43	824	22	390	28	610	435	8	17
44	846	21	418	29	582	427	7	16
45	867	21	447	29	553	420	8	15
46	888	21	476	29	524	412	7	14
47	909	22	505	28	495	405	8	13
48	931	21	533	29	467	397	7	12
49	952	21	562	29	438	390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	10
51	994	21	619	29	381	375	8	9
52	9.71 015	21	648	29	352	367	7	8
53	036	22	677	29	323	360	8	7
54	058	21	706	28	294	352	8	6
55	079	21	734	29	266	344	7	5
56	100	21	763	28	237	337	8	4
57	121	21	791	29	209	329	7	3
58	142	21	820	29	180	322	8	2
59	163	21	849	28	151	314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	30	29
1″	0.5	0.5
2″	1.0	1.0
3″	1.5	1.5
4″	2.0	1.9
5″	2.5	2.4
6″	3.0	2.9
7″	3.5	3.4
8″	4.0	3.9
9″	4.5	4.4

1′	28
1″	0.5
2″	0.9
3″	1.4
4″	1.9
5″	2.3
6″	2.8
7″	3.3
8″	3.7
9″	4.2

1′	22
1″	0.4
2″	0.7
3″	1.1
4″	1.5
5″	1.8
6″	2.2
7″	2.6
8″	2.9
9″	3.3

1′	21
1″	0.4
2″	0.7
3″	1.1
4″	1.4
5″	1.8
6″	2.1
7″	2.5
8″	2.8
9″	3.2

1′	7	8
1″	0.12	0.13
2″	0.23	0 27
3″	0.35	0.40
4″	0.47	0.53
5″	0.58	0.67
6″	0.70	0.80
7″	0.82	0.93
8″	0.93	1.07
9″	1.05	1.20

59°

Logarithmic Functions

31°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	'
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	**60**
1	205	21	906	29	094	299	8	59
2	226	21	935	28	065	291	7	58
3	247	21	963	29	037	284	8	57
4	268	21	992	28	008	276	7	56
5	289	21	9.78 020	29	0.21 980	269	8	55
6	310	21	049	28	951	261	8	54
7	331	21	077	29	923	253	7	53
8	352	21	106	29	894	246	8	52
9	373	20	135	28	865	238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	**50**
11	414	21	192	28	808	223	8	49
12	435	21	220	29	780	215	8	48
13	456	21	249	28	751	207	7	47
14	477	21	277	29	723	200	8	46
15	498	21	306	28	694	192	8	45
16	519	20	334	29	666	184	7	44
17	539	21	363	28	637	177	8	43
18	560	21	391	28	609	169	8	42
19	581	21	419	29	581	161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	**40**
21	622	21	476	29	524	146	8	39
22	643	21	505	28	495	138	7	38
23	664	21	533	29	467	131	8	37
24	685	20	562	28	438	123	8	36
25	705	21	590	28	410	115	7	35
26	726	21	618	29	382	108	8	34
27	747	20	647	28	353	100	8	33
28	767	21	675	29	325	092	8	32
29	788	21	704	28	296	084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	**30**
31	829	21	760	29	240	069	8	29
32	850	20	789	28	211	061	8	28
33	870	21	817	28	183	053	7	27
34	891	20	845	29	155	046	8	26
35	911	21	874	28	126	038	8	25
36	932	20	902	28	098	030	8	24
37	952	21	930	29	070	022	8	23
38	973	21	959	28	041	014	7	22
39	994	20	987	28	013	007	8	21
40	9.72 014	20	9.79 015	28	0.20 985	9.92 999	8	**20**
41	034	21	043	29	957	991	8	19
42	055	20	072	28	928	983	7	18
43	075	21	100	28	900	976	8	17
44	096	20	128	28	872	968	8	16
45	116	21	156	29	844	960	8	15
46	137	20	185	28	815	952	8	14
47	157	20	213	28	787	944	8	13
48	177	21	241	28	759	936	7	12
49	198	20	269	28	731	929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	**10**
51	238	21	326	28	674	913	8	9
52	259	20	354	28	646	905	8	8
53	279	20	382	28	618	897	8	7
54	299	21	410	28	590	889	8	6
55	320	20	438	28	562	881	7	5
56	340	20	466	29	534	874	8	4
57	360	21	495	28	505	866	8	3
58	381	20	523	28	477	858	8	2
59	401	20	551	28	449	850	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		**0**
'	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

P. P.

1'	29
1″	0.5
2″	1.0
3″	1.5
4″	1.9
5″	2.4
6″	2.9
7″	3.4
8″	3.9
9″	4.4

1'	28
1″	0.5
2″	0.9
3″	1.4
4″	1.9
5″	2.3
6″	2.8
7″	3.3
8″	3.7
9″	4.2

1'	21
1″	0.4
2″	0.7
3″	1.1
4″	1.4
5″	1.8
6″	2.1
7″	2.5
8″	2.8
9″	3.2

1'	20
1″	0.3
2″	0.7
3″	1.0
4″	1.3
5″	1.7
6″	2.0
7″	2.3
8″	2.7
9″	3.0

1'	7	8
1″	0.12	0.13
2″	0.23	0.27
3″	0.35	0.40
4″	0.47	0.53
5″	0.58	0.67
6″	0.70	0.80
7″	0.82	0.93
8″	0.93	1.07
9″	1.05	1.20

58°

32°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	**60**
1	441	20	607	28	393̲	834	8	59
2	461	21	635	28	365̄	826	8	58
3	482	20	663	28	337	818	8	57
4	502	20	691	28	309	810	7	56
5	522	20	719	28	281	803	8	55
6	542	20	747	29	253	795̄	8	54
7	562	20	776	28	224	787	8	53
8	582	20	804	28	196	779	8	52
9	602	20	832	28	168	771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	**50**
11	643	20	888	28	112	755̄	8	49
12	663	20	916	28	084	747	8	48
13	683	20	944	28	056	739	8	47
14	703	20	972	28	028	731	8	46
15	723	20	9.80 000	28	000	723	8	45
16	743	20	028	28	0.19 972	715	8	44
17	763	20	056	28	944	707	8	43
18	783	20	084	28	916	699	8	42
19	803	20	112	28	888	691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	**40**
21	843	20	168	27	832̲	675	8	39
22	863	20	195	28	805̄	667	8	38
23	883	19	223	28	777	659	8	37
24	902	20	251	28	749	651	8	36
25	922	20	279	28	721	643	8	35
26	942	20	307	28	693̲	635	8	34
27	962	20	335	28	665̄	627	8	33
28	982	20	363	28	637	619	8	32
29	9.73 002	20	391	28	609	611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	**30**
31	041	20	447	27	553	595̄	8	29
32	061	20	474	28	526	587̄	8	28
33	081	20	502	28	498	579	8	27
34	101	20	530	28	470	571	8	26
35	121	19	558	28	442	563	8	25
36	140	20	586	28	414	555̄	9	24
37	160	20	614	28	386	546	8	23
38	180	20	642	27	358	538	8	22
39	200	19	669	28	331	530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	**20**
41	239	20	725̄	28	275	514	8	19
42	259	19	753	28	247	506	8	18
43	278	20	781	27	219	498	8	17
44	298	20	808	28	192	490	8	16
45	318	19	836	28	164	482	9	15
46	337	20	864	28	136	473	8	14
47	357	20	892	27	108	465	8	13
48	377	19	919	28	081	457	8	12
49	396	20	947̲	28	053	449	8	11
50	9.73 416	19	9.80 975̄	28	0.19 025	9.92 441	8	**10**
51	435̲	20	9.81 003	27	0.18 997	433̲	8	9
52	455̄	19	030	28	970	425̄	9	8
53	474	20	058	28	942	416	8	7
54	494	19	086	27	914	408	8	6
55	513	20	113	28	887	400	8	5
56	533	19	141	28	859	392	8	4
57	552	20	169	27	831	384	8	3
58	572	19	196	28	804	376	9	2
59	591	20	224	28	776	367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	29	28
1″	0.5	0.5
2″	1.0	0.9
3″	1.5	1.4
4″	1.9	1.9
5″	2.4	2.3
6″	2.9	2.8
7″	3.4	3.3
8″	3.9	3.7
9″	4.4	4.2

1′	27	21
1″	0.5	0.4
2″	0.9	0.7
3″	1.4	1.1
4″	1.8	1.4
5″	2.3	1.8
6″	2.7	2.1
7″	3.2	2.5
8″	3.6	2.8
9″	4.1	3.2

1′	20	19
1″	0.3	0.3
2″	0.7	0.6
3″	1.0	1.0
4″	1.3	1.3
5″	1.7	1.6
6″	2.0	1.9
7″	2.3	2.2
8″	2.7	2.5
9″	3.0	2.9

1′	7
1″	0.12
2″	0.23
3″	0.35
4″	0.47
5″	0.58
6″	0.70
7″	0.82
8″	0.93
9″	1.05

1′	8	9
1″	0.13	0.15
2″	0.27	0.30
3″	0.40	0.45
4″	0.53	0.60
5″	0.67	0.75
6″	0.80	0.90
7″	0.93	1.05
8″	1.07	1.20
9″	1.20	1.35

57°

Logarithmic Functions

33°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	60
1	630	20	279	28	721	351	8	59
2	650	19	307	28	693	343	8	58
3	669	20	335	27	665	335	9	57
4	689	19	362	28	638	326	8	56
5	708	19	390	28	610	318	8	55
6	727	20	418	27	582	310	8	54
7	747	19	445	28	555	302	9	53
8	766	19	473	27	527	293	8	52
9	785	20	500	28	500	285	8	51
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	50
11	824	19	556	27	444	269	9	49
12	843	20	583	28	417	260	8	48
13	863	19	611	27	389	252	8	47
14	882	19	638	28	362	244	9	46
15	901	20	666	27	334	235	8	45
16	921	19	693	28	307	227	8	44
17	940	19	721	27	279	219	8	43
18	959	19	748	28	252	211	9	42
19	978	19	776	27	224	202	8	41
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	40
21	9.74 017	19	831	27	169	186	9	39
22	036	19	858	28	142	177	8	38
23	055	19	886	27	114	169	8	37
24	074	19	913	28	087	161	9	36
25	093	20	941	27	059	152	8	35
26	113	19	968	28	032	144	8	34
27	132	19	996	27	004	136	9	33
28	151	19	9.82 023	28	0.17 977	127	8	32
29	170	19	051	27	949	119	8	31
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	30
31	208	19	106	27	894	102	8	29
32	227	19	133	28	867	094	8	28
33	246	19	161	27	839	086	9	27
34	265	19	188	27	812	077	8	26
35	284	19	215	28	785	069	9	25
36	303	19	243	27	757	060	8	24
37	322	19	270	28	730	052	8	23
38	341	19	298	27	702	044	9	22
39	360	19	325	27	675	035	8	21
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	9	20
41	398	19	380	27	620	018	8	19
42	417	19	407	28	593	010	8	18
43	436	19	435	27	565	002	9	17
44	455	19	462	27	538	9.91 993	8	16
45	474	19	489	28	511	985	9	15
46	493	19	517	27	483	976	8	14
47	512	19	544	27	456	968	9	13
48	531	18	571	28	429	959	8	12
49	549	19	599	27	401	951	9	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	8	10
51	587	19	653	28	347	934	9	9
52	606	19	681	27	319	925	8	8
53	625	19	708	27	292	917	9	7
54	644	18	735	27	265	908	8	6
55	662	19	762	28	238	900	9	5
56	681	19	790	27	210	891	8	4
57	700	19	817	27	183	883	9	3
58	719	18	844	27	156	874	8	2
59	737	19	871	28	129	866	9	1
60	9.74 756		9.82 899		0.17 101	9.91 857		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	28	27
1″	0.5	0.5
2″	0.9	0.9
3″	1.4	1.4
4″	1.9	1.8
5″	2.3	2.3
6″	2.8	2.7
7″	3.3	3.2
8″	3.7	3.6
9″	4.2	4.1

1′	20
1″	0.3
2″	0.7
3″	1.0
4″	1.3
5″	1.7
6″	2.0
7″	2.3
8″	2.7
9″	3.0

1′	19
1″	0.3
2″	0.6
3″	1.0
4″	1.3
5″	1.6
6″	1.9
7″	2.2
8″	2.5
9″	2.9

1′	18
1″	0.3
2″	0.6
3″	0.9
4″	1.2
5″	1.5
6″	1.8
7″	2.1
8″	2.4
9″	2.7

1′	8	9
1″	0.13	0.15
2″	0.27	0.30
3″	0.40	0.45
4″	0.53	0.60
5″	0.67	0.75
6″	0.80	0.90
7″	0.93	1.05
8″	1.07	1.20
9″	1.20	1.35

56°

34°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	60
1	77$\bar{5}$	19	926	27	074	849	9	59
2	794	18	953	27	047	840	8	58
3	812	19	980	28	020	832	9	57
4	831	19	9.83 008	27	0.16 992	823	8	56
5	850	18	03$\bar{5}$	27	965	81$\bar{5}$	9	55
6	868	19	062	27	938	806	8	54
7	887	19	089	28	911	798	9	53
8	906	18	117	27	883	789	8	52
9	924	19	144	27	856	781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	**50**
11	961	19	198	27	802	763	8	49
12	980	19	225	27	77$\bar{5}$	75$\bar{5}$	9	48
13	999	18	252	28	748	746	8	47
14	9.75 017	19	280	27	720	738	9	46
15	036	18	307	27	693	729	9	45
16	054	19	334	27	666	720	8	44
17	073	18	361	27	639	712	9	43
18	091	19	388	27	612	703	8	42
19	110	18	415	27	58$\bar{5}$	69$\bar{5}$	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	**40**
21	147	18	470	27	530	677	8	39
22	165	19	497	27	503	669	9	38
23	184	18	524	27	476	660	9	37
24	202	19	551	27	449	651	8	36
25	221	18	578	27	422	643	9	35
26	239	19	605	27	39$\bar{5}$	634	9	34
27	258	18	632	27	368	625	8	33
28	276	18	659	27	341	617	9	32
29	294	19	686	27	314	608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	**30**
31	331	19	740	28	260	591	9	29
32	350	18	768	27	232	582	9	28
33	368	18	79$\bar{5}$	27	205	573	8	27
34	386	19	822	27	178	56$\bar{5}$	9	26
35	40$\bar{5}$	18	849	27	151	556	9	25
36	423	18	876	27	124	547	9	24
37	441	18	903	27	097	538	8	23
38	459	19	930	27	070	530	9	22
39	478	18	957	27	043	521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	**20**
41	514	19	9.84 011	27	0.15 989	504	9	19
42	533	18	038	27	962	49$\bar{5}$	9	18
43	551	18	06$\bar{5}$	27	935	486	9	17
44	569	18	092	27	908	477	8	16
45	587	18	119	27	881	469	9	15
46	605	19	146	27	854	460	9	14
47	624	18	173	27	827	451	9	13
48	642	18	200	27	800	442	9	12
49	660	18	227	27	773	433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 42$\bar{5}$	9	**10**
51	696	18	280	27	720	416	9	9
52	714	19	307	27	693	407	9	8
53	733	18	334	27	666	398	9	7
54	751	18	361	27	639	389	8	6
55	769	18	388	27	612	381	9	5
56	787	18	415	27	58$\bar{5}$	372	9	4
57	80$\bar{5}$	18	442	27	558	363	9	3
58	823	18	469	27	531	354	9	2
59	841	18	496	27	504	345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	28	27
1″	0.5	0.5
2″	0.9	0.9
3″	1.4	1.4
4″	1.9	1.8
5″	2.3	2.3
6″	2.8	2.7
7″	3.3	3.2
8″	3.7	3.6
9″	4.2	4.1

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	19
1″	0.3
2″	0.6
3″	1.0
4″	1.3
5″	1.6
6″	1.9
7″	2.2
8″	2.5
9″	2.9

1′	18
1″	0.3
2″	0.6
3″	0.9
4″	1.2
5″	1.5
6″	1.8
7″	2.1
8″	2.4
9″	2.7

1′	8	9
1″	0.13	0.15
2″	0.27	0.30
3″	0.40	0.45
4″	0.53	0.60
5″	0.67	0.75
6″	0.80	0.90
7″	0.93	1.05
8″	1.07	1.20
9″	1.20	1.35

55°

Logarithmic Functions

35°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60
1	877	18	550	26	450	328	9	59
2	895	18	576	27	424	319	9	58
3	913	18	603	27	397	310	9	57
4	931	18	630	27	370	301	9	56
5	949	18	657	27	343	292	9	55
6	967	18	684	27	316	283	9	54
7	985	18	711	27	289	274	8	53
8	9.76 003	18	738	26	262	266	9	52
9	021	18	764	27	236	257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50
11	057	18	818	27	182	239	9	49
12	075	18	845	27	155	230	9	48
13	093	18	872	27	128	221	9	47
14	111	18	899	26	101	212	9	46
15	129	17	925	27	075	203	9	45
16	146	18	952	27	048	194	9	44
17	164	18	979	27	021	185	9	43
18	182	18	9.85 006	27	0.14 994	176	9	42
19	200	18	033	26	967	167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40
21	236	17	086	27	914	149	8	39
22	253	18	113	27	887	141	9	38
23	271	18	140	26	860	132	9	37
24	289	18	166	27	834	123	9	36
25	307	17	193	27	807	114	9	35
26	324	18	220	27	780	105	9	34
27	342	18	247	26	753	096	9	33
28	360	18	273	27	727	087	9	32
29	378	17	300	27	700	078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30
31	413	18	354	26	646	060	9	29
32	431	17	380	27	620	051	9	28
33	448	18	407	27	593	042	9	27
34	466	18	434	26	566	033	10	26
35	484	17	460	27	540	023	9	25
36	501	18	487	27	513	014	9	24
37	519	18	514	26	486	005	9	23
38	537	17	540	27	460	9.90 996	9	22
39	554	18	567	27	433	987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20
41	590	17	620	27	380	969	9	19
42	607	18	647	27	353	960	9	18
43	625	17	674	26	326	951	9	17
44	642	18	700	27	300	942	9	16
45	660	17	727	27	273	933	9	15
46	677	18	754	26	246	924	9	14
47	695	17	780	27	220	915	9	13
48	712	18	807	27	193	906	10	12
49	730	17	834	26	166	896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10
51	765	17	887	26	113	878	9	9
52	782	18	913	27	087	869	9	8
53	800	17	940	27	060	860	9	7
54	817	18	967	26	033	851	9	6
55	835	17	993	27	007	842	10	5
56	852	18	9.86 020	26	0.13 980	832	9	4
57	870	17	046	27	954	823	9	3
58	887	17	073	27	927	814	9	2
59	904	18	100	26	900	805	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	27	26
1″	0.5	0.4
2″	0.9	0.9
3″	1.4	1.3
4″	1.8	1.7
5″	2.3	2.2
6″	2.7	2.6
7″	3.2	3.0
8″	3.6	3.5
9″	4.1	3.9

1′	18
1″	0.3
2″	0.6
3″	0.9
4″	1.2
5″	1.5
6″	1.8
7″	2.1
8″	2.4
9″	2.7

1′	17
1″	0.3
2″	0.6
3″	0.9
4″	1.1
5″	1.4
6″	1.7
7″	2.0
8″	2.3
9″	2.6

1′	8
1″	0.13
2″	0.27
3″	0.40
4″	0.53
5″	0.67
6″	0.80
7″	0.93
8″	1.07
9″	1.20

1′	9	10
1″	0.15	0.17
2″	0.30	0.33
3″	0.45	0.50
4″	0.60	0.67
5″	0.75	0.83
6″	0.90	1.00
7″	1.05	1.17
8″	1.20	1.33
9″	1.35	1.50

54°

36°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	**60**
1	939	18	153	26	847	787	10	59
2	957	17	179	27	821	777	9	58
3	974	17	206	26	794	768	9	57
4	991	18	232	27	768	759	9	56
5	9.77 009	17	259	26	741	750	9	55
6	026	17	285	27	715̄	741	10	54
7	043	18	312	26	688	731	9	53
8	061	17	338	27	662̲	722	9	52
9	078	17	365	27	635̄	713	9	51
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	10	**50**
11	112	18	418̲	27	582	694	9	49
12	130	17	445̄	26	555	685	9	48
13	147	17	471	27	529	676	9	47
14	164	17	498	26	502	667	10	46
15	181	18	524	27	476	657	9	45
16	199	17	551	26	449	648	9	44
17	216	17	577	26	423	639	9	43
18	233	17	603	27	397	630	10	42
19	250	18	630	26	370	620	9	41
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	**40**
21	285̄	17	683	26	317	602	10	39
22	302	17	709	27	291	592	9	38
23	319	17	736	26	264	583	9	37
24	336	17	762	27	238	574̲	9	36
25	353	17	789	26	211	565̄	10	35
26	370	17	815	27	185̄	555	9	34
27	387̲	18	842	26	158	546	9	33
28	405̄	17	868	26	132	537	10	32
29	422	17	894	27	106	527	9	31
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	**30**
31	456	17	947	27	053	509	10	29
32	473	17	974	26	026	499	9	28
33	490	17	9.87 000	27	000	490	10	27
34	507	17	027	26	0.12 973	480	9	26
35	524	17	053	26	947	471	9	25
36	541	17	079	27	921	462	10	24
37	558	17	106	26	894	452	9	23
38	575	17	132	26	868	443	9	22
39	592	17	158̲	27	842	434	10	21
40	9.77 609	17	9.87 185̄	26	0.12 815	9.90 424	9	**20**
41	626	17	211	27	789	415̄	10	19
42	643	17	238	26	762	405	9	18
43	660	17	264	26	736	396	10	17
44	677	17	290	27	710	386	9	16
45	694	17	317	26	683	377	9	15
46	711	17	343	26	657	368	10	14
47	728	16	369	27	631	358	9	13
48	744	17	396	26	604	349	10	12
49	761	17	422	26	578	339	9	11
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	**10**
51	795	17	475̄	26	525	320	9	9
52	812	17	501	26	499	311	10	8
53	829	17	527	27	473	301	9	7
54	846	16	554	26	446	292	10	6
55	862	17	580	26	420	282	9	5
56	879	17	606	27	394	273	10	4
57	896	17	633	26	367	263	9	3
58	913	17	659	26	341̲	254	10	2
59	930	16	685	26	315̄	244̲	9	1
60	9.77 946		9.87 711		0.12 289	9.90 235̄		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	27	26
1″	0.5	0.4
2″	0.9	0.9
3″	1.4	1.3
4″	1.8	1.7
5″	2.3	2.2
6″	2.7	2.6
7″	3 2	3.0
8″	3.6	3.5
9″	4.1	3.9

1′	18
1″	0.3
2″	0.6
3″	0.9
4″	1.2
5″	1.5
6″	1.8
7″	2.1
8″	2.4
9″	2.7

1′	17
1″	0.3
2″	0.6
3″	0.9
4″	1.1
5″	1.4
6″	1.7
7″	2.0
8″	2.3
9″	2.6

1′	16
1″	0.3
2″	0.5
3″	0.8
4″	1.1
5″	1.3
6″	1.6
7″	1.9
8″	2.1
9″	2.4

1′	9	10
1″	0.15	0.2
2″	0.30	0.3
3″	0.45	0.5
4″	0.60	0.7
5″	0.75	0.8
6″	0.90	1.0
7″	1.05	1.2
8″	1.20	1.3
9″	1.35	1.5

53°

37°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	60
1	963	17	738	26	262	225	9	59
2	980	17	764	26	236	216	10	58
3	997	16	790	27	210	206	9	57
4	9.78 013	17	817	26	183	197	10	56
5	030	17	843	26	157	187	9	55
6	047	16	869	26	131	178	10	54
7	063	17	895	27	105	168	9	53
8	080	17	922	26	078	159	10	52
9	097	16	948	26	052	149	10	51
10	9.78 113	17	9.87 974	26	0.12 026	9.90 139	9	50
11	130	17	9.88 000	27	000	130	10	49
12	147	16	027	26	0.11 973	120	9	48
13	163	17	053	26	947	111	10	47
14	180	17	079	26	921	101	10	46
15	197	16	105	26	895	091	9	45
16	213	17	131	27	869	082	10	44
17	230	16	158	26	842	072	9	43
18	246	17	184	26	816	063	10	42
19	263	17	210	26	790	053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	40
21	296	17	262	27	738	034	10	39
22	313	16	289	26	711	024	10	38
23	329	17	315	26	685	014	9	37
24	346	16	341	26	659	005	10	36
25	362	17	367	26	633	9.89 995	10	35
26	379	16	393	27	607	985	9	34
27	395	17	420	26	580	976	10	33
28	412	16	446	26	554	966	10	32
29	428	17	472	26	528	956	9	31
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	10	30
31	461	17	524	26	476	937	10	29
32	478	16	550	27	450	927	9	28
33	494	16	577	26	423	918	10	27
34	510	17	603	26	397	908	10	26
35	527	16	629	26	371	898	10	25
36	543	17	655	26	345	888	9	24
37	560	16	681	26	319	879	10	23
38	576	16	707	26	293	869	10	22
39	592	17	733	26	267	859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	20
41	625	17	786	26	214	840	10	19
42	642	16	812	26	188	830	10	18
43	658	16	838	26	162	820	10	17
44	674	17	864	26	136	810	9	16
45	691	16	890	26	110	801	10	15
46	707	16	916	26	084	791	10	14
47	723	16	942	26	058	781	10	13
48	739	17	968	26	032	771	10	12
49	756	16	994	26	006	761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	10
51	788	17	046	27	954	742	10	9
52	805	16	073	26	927	732	10	8
53	821	16	099	26	901	722	10	7
54	837	16	125	26	875	712	10	6
55	853	16	151	26	849	702	9	5
56	869	17	177	26	823	693	10	4
57	886	16	203	26	797	683	10	3
58	902	16	229	26	771	673	10	2
59	918	16	255	26	745	663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	27
1″	0.5
2″	0.9
3″	1.4
4″	1.8
5″	2.3
6″	2.7
7″	3.2
8″	3.6
9″	4.1

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	17
1″	0.3
2″	0.6
3″	0.9
4″	1.1
5″	1.4
6″	1.7
7″	2.0
8″	2.3
9″	2.6

1′	16
1″	0.3
2″	0.5
3″	0.8
4″	1.1
5″	1.3
6″	1.6
7″	1.9
8″	2.1
9″	2.4

1′	9	10
1″	0.15	0.2
2″	0.30	0.3
3″	0.45	0.5
4″	0.60	0.7
5″	0.75	0.8
6″	0.90	1.0
7″	1.05	1.2
8″	1.20	1.3
9″	1.35	1.5

52°

38°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60
1	950	17	307	26	693	643	10	59
2	967	16	333	26	667	633	9	58
3	983	16	359	26	641	624	10	57
4	999	16	385	26	615	614	10	56
5	9.79 015	16	411	26	589	604	10	55
6	031	16	437	26	563	594	10	54
7	047	16	463	26	537	584	10	53
8	063	16	489	26	511	574	10	52
9	079	16	515	26	485	564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	**50**
11	111	17	567	26	433	544	10	49
12	128	16	593	26	407	534	10	48
13	144	16	619	26	381	524	10	47
14	160	16	645	26	355	514	10	46
15	176	16	671	26	329	504	9	45
16	192	16	697	26	303	495	10	44
17	208	16	723	26	277	485	10	43
18	224	16	749	26	251	475	10	42
19	240	16	775	26	225	465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	**40**
21	272	16	827	26	173	445	10	39
22	288	16	853	26	147	435	10	38
23	304	15	879	26	121	425	10	37
24	319	16	905	26	095	415	10	36
25	335	16	931	26	069	405	10	35
26	351	16	957	26	043	395	10	34
27	367	16	983	26	017	385	10	33
28	383	16	9.90 009	26	0.09 991	375	11	32
29	399	16	035	26	965	364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	**30**
31	431	16	086	26	914	344	10	29
32	447	16	112	26	888	334	10	28
33	463	15	138	26	862	324	10	27
34	478	16	164	26	836	314	10	26
35	494	16	190	26	810	304	10	25
36	510	16	216	26	784	294	10	24
37	526	16	242	26	758	284	10	23
38	542	16	268	26	732	274	10	22
39	558	15	294	26	706	264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	**20**
41	589	16	346	25	654	244	11	19
42	605	16	371	26	629	233	10	18
43	621	15	397	26	603	223	10	17
44	636	16	423	26	577	213	10	16
45	652	16	449	26	551	203	10	15
46	668	16	475	26	525	193	10	14
47	684	15	501	26	499	183	10	13
48	699	16	527	26	473	173	11	12
49	715	16	553	25	447	162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	**10**
51	746	16	604	26	396	142	10	9
52	762	16	630	26	370	132	10	8
53	778	15	656	26	344	122	10	7
54	793	16	682	26	318	112	11	6
55	809	16	708	26	292	101	10	5
56	825	15	734	25	266	091	10	4
57	840	16	759	26	241	081	10	3
58	856	16	785	26	215	071	11	2
59	872	15	811	26	189	060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26	25
1″	0.4	0.4
2″	0.9	0.8
3″	1.3	1.3
4″	1.7	1.7
5″	2.2	2.1
6″	2.6	2.5
7″	3.0	2.9
8″	3.5	3.3
9″	3.9	3.8

1′	17	16
1″	0.3	0.3
2″	0.6	0.5
3″	0.9	0.8
4″	1.1	1.1
5″	1.4	1.3
6″	1.7	1.6
7″	2.0	1.9
8″	2.3	2.1
9″	2.6	2.4

1′	15
1″	0.3
2″	0.5
3″	0.8
4″	1.0
5″	1.3
6″	1.5
7″	1.8
8″	2.0
9″	2.3

1′	9
1″	0.15
2″	0.30
3″	0.45
4″	0.60
5″	0.75
6″	0.90
7″	1.05
8″	1.20
9″	1.35

1′	10	11
1″	0.2	0.2
2″	0.3	0.4
3″	0.5	0.6
4″	0.7	0.7
5″	0.8	0.9
6″	1.0	1.1
7″	1.2	1.3
8″	1.3	1.5
9″	1.5	1.7

51°

39°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	**60**
1	903	15	863	26	137	040	10	59
2	918	16	889	25	111	030	10	58
3	934	16	914	26	086	020	11	57
4	950	15	940	26	060	009	10	56
5	965	16	966	26	034	9.88 999	10	55
6	981	15	992	26	008	989	11	54
7	996	16	9.91 018	25	0.08 982	978	10	53
8	9.80 012	15	043	26	957	968	10	52
9	027	16	069	26	931	958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	**50**
11	058	16	121	26	879	937	10	49
12	074	15	147	25	853	927	10	48
13	089	16	172	26	828	917	11	47
14	105	15	198	26	802	906	10	46
15	120	16	224	26	776	896	10	45
16	136	15	250	26	750	886	11	44
17	151	15	276	25	724	875	10	43
18	166	16	301	26	699	865	10	42
19	182	15	327	26	673	855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	**40**
21	213	15	379	25	621	834	10	39
22	228	16	404	26	596	824	11	38
23	244	15	430	26	570	813	10	37
24	259	15	456	26	544	803	10	36
25	274	16	482	25	518	793	11	35
26	290	15	507	26	493	782	10	34
27	305	15	533	26	467	772	11	33
28	320	16	559	26	441	761	10	32
29	336	15	585	25	415	751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	**30**
31	366	16	636	26	364	730	10	29
32	382	15	662	26	338	720	11	28
33	397	15	688	25	312	709	10	27
34	412	16	713	26	287	699	11	26
35	428	15	739	26	261	688	10	25
36	443	15	765	26	235	678	10	24
37	458	15	791	25	209	668	11	23
38	473	16	816	26	184	657	10	22
39	489	15	842	26	158	647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	**20**
41	519	15	893	26	107	626	11	19
42	534	16	919	26	081	615	10	18
43	550	15	945	26	055	605	11	17
44	565	15	971	25	029	594	10	16
45	580	15	996	26	004	584	11	15
46	595	15	9.92 022	26	0.07 978	573	10	14
47	610	15	048	25	952	563	11	13
48	625	16	073	26	927	552	10	12
49	641	15	099	26	901	542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	**10**
51	671	15	150	26	850	521	11	9
52	686	15	176	26	824	510	11	8
53	701	15	202	25	798	499	10	7
54	716	15	227	26	773	489	11	6
55	731	15	253	26	747	478	10	5
56	746	16	279	25	721	468	11	4
57	762	15	304	26	696	457	10	3
58	777	15	330	26	670	447	11	2
59	792	15	356	25	644	436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	25
1″	0.4
2″	0.8
3″	1.3
4″	1.7
5″	2.1
6″	2.5
7″	2.9
8″	3.3
9″	3.8

1′	16
1″	0.3
2″	0.5
3″	0.8
4″	1.1
5″	1.3
6″	1.6
7″	1.9
8″	2.1
9″	2.4

1′	15
1″	0.3
2″	0.5
3″	0.8
4″	1.0
5″	1.3
6″	1.5
7″	1.8
8″	2.0
9″	2.3

1′	10	11
1″	0.2	0.2
2″	0.3	0.4
3″	0.5	0.6
4″	0.7	0.7
5″	0.8	0.9
6″	1.0	1.1
7″	1.2	1.3
8″	1.3	1.5
9″	1.5	1.7

50°

40°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.80 807	15	9.92 381	26	0.07 619	9.88 425	10	60
1	822	15	407	26	593	415	11	59
2	837	15	433	25	567	404	10	58
3	852	15	458	26	542	394	11	57
4	867	15	484	26	516	383	11	56
5	882	15	510	25	490	372	10	55
6	897	15	535	26	465	362	11	54
7	912	15	561	26	439	351	11	53
8	927	15	587	25	413	340	10	52
9	942	15	612	26	388	330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	50
11	972	15	663	26	337	308	10	49
12	987	15	689	26	311	298	11	48
13	9.81 002	15	715	25	285	287	11	47
14	017	15	740	26	260	276	10	46
15	032	15	766	26	234	266	11	45
16	047	14	792	25	208	255	11	44
17	061	15	817	26	183	244	10	43
18	076	15	843	25	157	234	11	42
19	091	15	868	26	132	223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	40
21	121	15	920	25	080	201	10	39
22	136	15	945	26	055	191	11	38
23	151	15	971	25	029	180	11	37
24	166	14	996	26	004	169	11	36
25	180	15	9.93 022	26	0.06 978	158	10	35
26	195	15	048	25	952	148	11	34
27	210	15	073	26	927	137	11	33
28	225	15	099	25	901	126	11	32
29	240	14	124	26	876	115	10	31
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	30
31	269	15	175	26	825	094	11	29
32	284	15	201	26	799	083	11	28
33	299	15	227	25	773	072	11	27
34	314	14	252	26	748	061	10	26
35	328	15	278	25	722	051	11	25
36	343	15	303	26	697	040	11	24
37	358	14	329	25	671	029	11	23
38	372	15	354	26	646	018	11	22
39	387	15	380	26	620	007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	20
41	417	14	431	26	569	985	10	19
42	431	15	457	25	543	975	11	18
43	446	15	482	26	518	964	11	17
44	461	14	508	25	492	953	11	16
45	475	15	533	26	467	942	11	15
46	490	15	559	25	441	931	11	14
47	505	14	584	26	416	920	11	13
48	519	15	610	26	390	909	11	12
49	534	15	636	25	364	898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	10
51	563	15	687	25	313	877	11	9
52	578	14	712	26	288	866	11	8
53	592	15	738	25	262	855	11	7
54	607	15	763	26	237	844	11	6
55	622	14	789	25	211	833	11	5
56	636	15	814	26	186	822	11	4
57	651	14	840	25	160	811	11	3
58	665	15	865	26	135	800	11	2
59	680	14	891	25	109	789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	25
1″	0.4
2″	0.8
3″	1.3
4″	1.7
5″	2.1
6″	2.5
7″	2.9
8″	3.3
9″	3.8

1″	15
1″	0.3
2″	0.5
3″	0.8
4″	1.0
5″	1.3
6″	1.5
7″	1.8
8″	2.0
9″	2.3

1′	14
1″	0.2
2″	0.5
3″	0.7
4″	0.9
5″	1.2
6″	1.4
7″	1.6
8″	1.9
9″	2.1

1′	10	11
1″	0.2	0.2
2″	0.3	0.4
3″	0.5	0.6
4″	0.7	0.7
5″	0.8	0.9
6″	1.0	1.1
7″	1.2	1.3
8″	1.3	1.5
9″	1.5	1.7

49°

41°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60
1	709	14	942	25	058	767	11	59
2	723	15	967	26	033	756	11	58
3	738	14	993	25	007	745	11	57
4	752	15	9.94 018	26	0.05 982	734	11	56
5	767	14	044	25	956	723	11	55
6	781	15	069	26	931	712	11	54
7	796	14	095	25	905	701	11	53
8	810	15	120	26	880	690	11	52
9	825	14	146	25	854	679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50
11	854	14	197	25	803	657	11	49
12	868	14	222	26	778	646	11	48
13	882	15	248	25	752	635	11	47
14	897	14	273	26	727	624	11	46
15	911	15	299	25	701	613	12	45
16	926	14	324	26	676	601	11	44
17	940	15	350	25	650	590	11	43
18	955	14	375	26	625	579	11	42
19	969	14	401	25	599	568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40
21	998	14	452	25	548	546	11	39
22	9.82 012	14	477	26	523	535	11	38
23	026	15	503	25	497	524	11	37
24	041	14	528	26	472	513	12	36
25	055	14	554	25	446	501	11	35
26	069	15	579	25	421	490	11	34
27	084	14	604	26	396	479	11	33
28	098	14	630	25	370	468	11	32
29	112	14	655	26	345	457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	30
31	141	14	706	26	294	434	11	29
32	155	14	732	25	268	423	11	28
33	169	15	757	26	243	412	11	27
34	184	14	783	25	217	401	11	26
35	198	14	808	26	192	390	12	25
36	212	14	834	25	166	378	11	24
37	226	14	859	25	141	367	11	23
38	240	15	884	26	116	356	11	22
39	255	14	910	25	090	345	11	21
40	9.82 269	14	9.94 935	26	0.05 065	9.87 334	12	20
41	283	14	961	25	039	322	11	19
42	297	14	986	26	014	311	11	18
43	311	15	9.95 012	25	0.04 988	300	12	17
44	326	14	037	25	963	288	11	16
45	340	14	062	26	938	277	11	15
46	354	14	088	25	912	266	11	14
47	368	14	113	26	887	255	12	13
48	382	14	139	25	861	243	11	12
49	396	14	164	26	836	232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	10
51	424	15	215	25	785	209	11	9
52	439	14	240	26	760	198	11	8
53	453	14	266	25	734	187	12	7
54	467	14	291	26	709	175	11	6
55	481	14	317	25	683	164	11	5
56	495	14	342	26	658	153	12	4
57	509	14	368	25	632	141	11	3
58	523	14	393	25	607	130	11	2
59	537	14	418	26	582	119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26	25	15	14
1″	0.4	0.4	0.3	0.2
2″	0.9	0.8	0.5	0.5
3″	1.3	1.3	0.8	0.7
4″	1.7	1.7	1.0	0.9
5″	2.2	2.1	1.3	1.2
6″	2.6	2.5	1.5	1.4
7″	3.0	2.9	1.8	1.6
8″	3.5	3.3	2.0	1.9
9″	3.9	3.8	2.3	2.1

1′	11	12
1″	0.2	0.2
2″	0.4	0.4
3″	0.6	0.6
4″	0.7	0.8
5″	0.9	1.0
6″	1.1	1.2
7″	1.3	1.4
8″	1.5	1.6
9″	1.7	1.8

48°

42°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	60
1	565	14	469	26	531	096	11	59
2	579	14	495̄	25	505	085̄	12	58
3	593	14	520	25	480	073	11	57
4	607	14	545	26	455̄	062	12	56
5	621	14	571	25	429	050	11	55
6	635	14	596	26	404	039	11	54
7	649	14	622	25	378	028	12	53
8	663	14	647	25	353	016	11	52
9	677	14	672	26	328	005̄	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	**50**
11	705̄	14	723	25	277	982	12	49
12	719	14	748	26	252	970	11	48
13	733	14	774	25	226	959	12	47
14	747	14	799	26	201	947	11	46
15	761	14	825̄	25	175	936	12	45
16	775̄	13	850	25	150	924	11	44
17	788	14	875	26	125̄	913	11	43
18	802	14	901	25	099	902	12	42
19	816	14	926	26	074	890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	**40**
21	844	14	977	25	023	867	12	39
22	858	14	9.96 002	26	0.03 998	855	11	38
23	872	13	028	25	972	844	12	37
24	885	14	053	25	947	832	11	36
25	899	14	078	26	922	821	12	35
26	913	14	104	25	896	809	11	34
27	927	14	129	26	871	798	12	33
28	941	14	155̄	25	845	786	11	32
29	955̄	13	180	25	820	775̄	12	31
30	9.82 968	14	9.96 205	26	0.03 795̄	9.86 763	11	**30**
31	982	14	231	25	769	752	12	29
32	996	14	256	25	744	740	12	28
33	9.83 010	13	281	26	719	728	11	27
34	023	14	307	25	693	717	12	26
35	037	14	332	25	668	705	11	25
36	051	14	357	26	643	694	12	24
37	065̄	13	383	25	617	682	12	23
38	078	14	408	25	592	670	11	22
39	092	14	433	26	567	659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	**20**
41	120	13	484	26	516	635	11	19
42	133	14	510	25	490	624	12	18
43	147	14	535̄	25	465	612	12	17
44	161	13	560	26	440	600	11	16
45	174	14	586	25	414	589	12	15
46	188	14	611	25	389	577	12	14
47	202	13	636	26	364	565	11	13
48	215	14	662	25	338	554	12	12
49	229	13	687	25	313	542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	**10**
51	256	14	738	25	262	518	11	9
52	270	13	763	25	237	507	12	8
53	283	14	788	26	212	495	12	7
54	297	13	814	25	186	483	11	6
55	310	14	839	25	161	472	12	5
56	324	14	864	26	136	460	12	4
57	338	13	890	25	110	448	12	3
58	351	14	915̄	25	085	436	11	2
59	365̄	13	940	26	060	425̄	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	25
1″	0.4
2″	0.8
3″	1.3
4″	1.7
5″	2.1
6″	2.5
7″	2.9
8″	3.3
9″	3.8

1′	14
1″	0.2
2″	0.5
3″	0.7
4″	0.9
5″	1.2
6″	1.4
7″	1.6
8″	1.9
9″	2.1

1′	13
1″	0.2
2″	0.4
3″	0.7
4″	0.9
5″	1.1
6″	1.3
7″	1.5
8″	1.7
9″	2.0

1′	12	11
1″	0.2	0.2
2″	0.4	0.4
3″	0.6	0.6
4″	0.8	0.7
5″	1.0	0.9
6″	1.2	1.1
7″	1.4	1.3
8″	1.6	1.5
9″	1.8	1.7

47°

43°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	392	13	991	25	009	401	12	59
2	405	14	9.97 016	26	0.02 984	389	12	58
3	419	13	042	25	958	377	11	57
4	432	14	067	25	933	366	12	56
5	446	13	092	26	908	354	12	55
6	459	14	118	25	882	342	12	54
7	473	13	143	25	857	330	12	53
8	486	14	168	25	832	318	12	52
9	500	13	193	26	807	306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	527	13	244	25	756	283	12	49
12	540	14	269	26	731	271	12	48
13	554	13	295	25	705	259	12	47
14	567	14	320	25	680	247	12	46
15	581	13	345	26	655	235	12	45
16	594	14	371	25	629	223	12	44
17	608	13	396	25	604	211	11	43
18	621	13	421	26	579	200	12	42
19	634	14	447	25	553	188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	661	13	497	26	503	164	12	39
22	674	14	523	25	477	152	12	38
23	688	13	548	25	452	140	12	37
24	701	14	573	25	427	128	12	36
25	715	13	598	26	402	116	12	35
26	728	13	624	25	376	104	12	34
27	741	14	649	25	351	092	12	33
28	755	13	674	26	326	080	12	32
29	768	13	700	25	300	068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	795	13	750	26	250	044	12	29
32	808	13	776	25	224	032	12	28
33	821	13	801	25	199	020	12	27
34	834	14	826	25	174	008	12	26
35	848	13	851	26	149	9.85 996	12	25
36	861	13	877	25	123	984	12	24
37	874	13	902	25	098	972	12	23
38	887	14	927	26	073	960	12	22
39	901	13	953	25	047	948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	927	13	9.98 003	26	0.01 997	924	12	19
42	940	14	029	25	971	912	12	18
43	954	13	054	25	946	900	12	17
44	967	13	079	25	921	888	12	16
45	980	13	104	26	896	876	12	15
46	993	13	130	25	870	864	13	14
47	9.84 006	14	155	25	845	851	12	13
48	020	13	180	26	820	839	12	12
49	033	13	206	25	794	827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	059	13	256	25	744	803	12	9
52	072	13	281	26	719	791	12	8
53	085	13	307	25	693	779	13	7
54	098	14	332	25	668	766	12	6
55	112	13	357	26	643	754	12	5
56	125	13	383	25	617	742	12	4
57	138	13	408	25	592	730	12	3
58	151	13	433	25	567	718	12	2
59	164	13	458	26	542	706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26	25
1″	0.4	0.4
2″	0.9	0.8
3″	1.3	1.3
4″	1.7	1.7
5″	2.2	2.1
6″	2.6	2.5
7″	3.0	2.9
8″	3.5	3.3
9″	3.9	3.8

1′	14	13
1″	0.2	0.2
2″	0.5	0.4
3″	0.7	0.7
4″	0.9	0.9
5″	1.2	1.1
6″	1.4	1.3
7″	1.6	1.5
8″	1.9	1.7
9″	2.1	2.0

1′	12	11
1″	0.2	0.2
2″	0.4	0.4
3″	0.6	0.6
4″	0.8	0.7
5″	1.0	0.9
6″	1.2	1.1
7″	1.4	1.3
8″	1.6	1.5
9″	1.8	1.7

46°

44°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	**60**
1	190	13	509	25	491	681	12	59
2	203	13	534	26	466	669	12	58
3	216	13	560̲	25	440	657̲	12	57
4	229	13	585̄	25	415	645̄	13	56
5	242	13	610	25	390	632	12	55
6	255	14	635	26	365̄	620	12	54
7	269	13	661	25	339	608	12	53
8	282̲	13	686	25	314	596	13	52
9	295̄	13	711	26	289	583	12	51
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	**50**
11	321	13	762	25	238	559	12	49
12	334	13	787	25	213	547	13	48
13	347	13	812	26	188	534	12	47
14	360	13	838	25	162	522	12	46
15	373	12	863	25	137	510	13	45
16	385	13	888	25	112	497̲	12	44
17	398	13	913	26	087	485̄	12	43
18	411	13	939	25	061	473	13	42
19	424	13	964	25	036	460	12	41
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	**40**
21	450	13	9.99 015̄	25	0.00 985	436	13	39
22	463	13	040	25	960̲	423	12	38
23	476	13	065	25	935̄	411	12	37
24	489	13	090	26	910	399	13	36
25	502	13	116	25	884	386	12	35
26	515̄	13	141	25	859	374	13	34
27	528	12	166	25	834	361	12	33
28	540	13	191	26	809	349	12	32
29	553	13	217	25	783	337	13	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	**30**
31	579	13	267	26	733	312	13	29
32	592̲	13	293	25	707	299	12	28
33	605̄	13	318	25	682	287	13	27
34	618	12	343	25	657	274	12	26
35	630	13	368	26	632	262	12	25
36	643	13	394	25	606	250	13	24
37	656	13	419	25	581	237̲	12	23
38	669	13	444	25	556	225̄	13	22
39	682	12	469̲	26	531	212	12	21
40	9.84 694	13	9.99 495̄	25	0.00 505	9.85 200	13	**20**
41	707	13	520	25	480̲	187̲	12	19
42	720	13	545	25	455̄	175̄	13	18
43	733	12	570	26	430	162	12	17
44	745	13	596	25	404	150	13	16
45	758	13	621	25	379	137	12	15
46	771	13	646	26	354	125̄	13	14
47	784	12	672	25	328	112	12	13
48	796	13	697	25	303	100	13	12
49	809	13	722	25	278	087	13	11
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	**10**
51	835̄	12	773	25	227	062	13	9
52	847	13	798	25	202	049	12	8
53	860	13	823	25	177	037	13	7
54	873	12	848	26	152	024	12	6
55	885	13	874	25	126	012	13	5
56	898	13	899	25	101	9.84 999	13	4
57	911	12	924	25	076	986	12	3
58	923	13	949̲	26	051	974	13	2
59	936	13	975̄	25	025	961	12	1
60	9.84 949		0.00 000		0.00 000	9.84 949		**0**
′	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

P. P.

1′	26
1″	0.4
2″	0.9
3″	1.3
4″	1.7
5″	2.2
6″	2.6
7″	3.0
8″	3.5
9″	3.9

1′	25
1″	0.4
2″	0.8
3″	1.3
4″	1.7
5″	2.1
6″	2.5
7″	2.9
8″	3.3
9″	3.8

1′	14
1″	0.2
2″	0.5
3″	0.7
4″	0.9
5″	1.2
6″	1.4
7″	1.6
8″	1.9
9″	2.1

1′	13
1″	0.2
2″	0.4
3″	0.7
4″	0.9
5″	1.1
6″	1.3
7″	1.5
8″	1.7
9″	2.0

1′	12
1″	0.2
2″	0.4
3″	0.6
4″	0.8
5″	1.0
6″	1.2
7″	1.4
8″	1.6
9″	1.8

45°

NATURAL
TRIGONOMETRIC FUNCTIONS

Natural Sines and Cosines

′	0°		1°		2°		3°		4°		′
	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.00000	1.00000	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	**60**
1	029	000	774	984	519	938	263	861	.07005	754	59
2	058	000	803	984	548	937	292	860	034	752	58
3	087	000	832	983	577	936	321	858	063	750	57
4	116	000	862	983	606	935	350	857	092	748	56
5	145	000	891	982	635	934	379	855	121	746	55
6	175	000	920	982	664	933	408	854	150	744	54
7	204	000	949	981	693	932	437	852	179	742	53
8	233	000	978	980	723	931	466	851	208	740	52
9	262	000	.02007	980	752	930	495	849	237	738	51
10	.00291	1.00000	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	**50**
11	320	.99999	065	979	810	927	553	846	295	734	49
12	349	999	094	978	839	926	582	844	324	731	48
13	378	999	123	977	868	925	611	842	353	729	47
14	407	999	152	977	897	924	640	841	382	727	46
15	436	999	181	976	926	923	669	839	411	725	45
16	465	999	211	976	955	922	698	838	440	723	44
17	495	999	240	975	984	921	727	836	469	721	43
18	524	999	269	974	.04013	919	756	834	498	719	42
19	553	998	298	974	042	918	785	833	527	716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	**40**
21	611	998	356	972	100	916	844	829	585	712	39
22	640	998	385	972	129	915	873	827	614	710	38
23	669	998	414	971	159	913	902	826	643	708	37
24	698	998	443	970	188	912	931	824	672	705	36
25	727	997	472	969	217	911	960	822	701	703	35
26	756	997	501	969	246	910	989	821	730	701	34
27	785	997	530	968	275	909	.06018	819	759	699	33
28	814	997	560	967	304	907	047	817	788	696	32
29	844	996	589	966	333	906	076	815	817	694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	**30**
31	902	996	647	965	391	904	134	812	875	689	29
32	931	996	676	964	420	902	163	810	904	687	28
33	960	995	705	963	449	901	192	808	933	685	27
34	989	995	734	963	478	900	221	806	962	683	26
35	.01018	995	763	962	507	898	250	804	991	680	25
36	047	995	792	961	536	897	279	803	.08020	678	24
37	076	994	821	960	565	896	308	801	049	676	23
38	105	994	850	959	594	894	337	799	078	673	22
39	134	994	879	959	623	893	366	797	107	671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	**20**
41	193	993	938	957	682	890	424	793	165	666	19
42	222	993	967	956	711	889	453	792	194	664	18
43	251	992	996	955	740	888	482	790	223	661	17
44	280	992	.03025	954	769	886	511	788	252	659	16
45	309	991	054	953	798	885	540	786	281	657	15
46	338	991	083	952	827	883	569	784	310	654	14
47	367	991	112	952	856	882	598	782	339	652	13
48	396	990	141	951	885	881	627	780	368	649	12
49	425	990	170	950	914	879	656	778	397	647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	**10**
51	483	989	228	948	972	876	714	774	455	642	9
52	513	989	257	947	.05001	875	743	772	484	639	8
53	542	988	286	946	030	873	773	770	513	637	7
54	571	988	316	945	059	872	802	768	542	635	6
55	600	987	345	944	088	870	831	766	571	632	5
56	629	987	374	943	117	869	860	764	600	630	4
57	658	986	403	942	146	867	889	762	629	627	3
58	687	986	432	941	175	866	918	760	658	625	2
59	716	985	461	940	205	864	947	758	687	622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	**0**
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	
′	89°		88°		87°		86°		85°		′

′	0°		1°		2°		3°		4°		′
	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	
0	.0000	Infinite	.0175	57.2900	.0349	28.6363	.0524	19.0811	.0699	14.3007	**60**
1	03	3437.75	77	56.3506	52	3994	27	18.9755	.0702	2411	59
2	06	1718.87	80	55.4415	55	1664	30	8711	05	1821	58
3	09	1145.92	83	54.5613	58	27.9372	33	7678	08	1235	57
4	12	859.436	86	53.7086	61	7117	36	6656	11	0655	56
5	15	687.549	89	52.8821	64	4899	39	5645	14	0079	55
6	17	572.957	92	0807	67	2715	42	4645	17	13.9507	54
7	20	491.106	95	51.3032	70	0566	44	3655	20	8940	53
8	23	429.718	98	50.5485	73	26.8450	47	2677	23	8378	52
9	26	381.971	.0201	49.8157	75	6367	50	1708	26	7821	51
10	.0029	343.774	.0204	49.1039	.0378	26.4316	.0553	18.0750	.0729	13.7267	**50**
11	32	312.521	07	48.4121	81	2296	56	17.9802	31	6719	49
12	35	286.478	09	47.7395	84	0307	59	8863	34	6174	48
13	38	264.441	12	0853	87	25.8348	62	7934	37	5634	47
14	41	245.552	15	46.4489	90	6418	65	7015	40	5098	46
15	44	229.182	18	45.8294	93	4517	68	6106	43	4566	45
16	47	214.858	21	2261	96	2644	71	5205	46	4039	44
17	49	202.219	24	44.6386	99	0798	74	4314	49	3515	43
18	52	190.984	27	0661	.0402	24.8978	77	3432	52	2996	42
19	55	180.932	30	43.5081	05	7185	80	2558	55	2480	41
20	.0058	171.885	.0233	42.9641	.0407	24.5418	.0582	17.1693	.0758	13.1969	**40**
21	61	163.700	36	4335	10	3675	85	0837	61	1461	39
22	64	156.259	39	41.9158	13	1957	88	16.9990	64	0958	38
23	67	149.465	41	4106	16	0263	91	9150	67	0458	37
24	70	143.237	44	40.9174	19	23.8593	94	8319	69	12.9962	36
25	73	137.507	47	4358	22	6945	97	7496	72	9469	35
26	76	132.219	50	39.9655	25	5321	.0600	6681	75	8981	34
27	79	127.321	53	5059	28	3718	03	5874	78	8496	33
28	81	122.774	56	0568	31	2137	06	5075	81	8014	32
29	84	118.540	59	38.6177	34	0577	09	4283	84	7536	31
30	.0087	114.589	.0262	38.1885	.0437	22.9038	.0612	16.3499	.0787	12.7062	**30**
31	90	110.892	65	37.7686	40	7519	15	2722	90	6591	29
32	93	107.426	68	3579	42	6020	17	1952	93	6124	28
33	96	104.171	71	36.9560	45	4541	20	1190	96	5660	27
34	99	101.107	74	5627	48	3081	23	0435	99	5199	26
35	.0102	98.2179	76	1776	51	1640	26	15.9687	.0802	4742	25
36	05	95.4895	79	35.8006	54	0217	29	8945	05	4288	24
37	08	92.9085	82	4313	57	21.8813	32	8211	08	3838	23
38	11	90.4633	85	0695	60	7426	35	7483	10	3390	22
39	13	88.1436	88	34.7151	63	6056	38	6762	13	2946	21
40	.0116	85.9398	.0291	34.3678	.0466	21.4704	.0641	15.6048	.0816	12.2505	**20**
41	19	83.8435	94	0273	69	3369	44	5340	19	2067	19
42	22	81.8470	97	33.6935	72	2049	47	4638	22	1632	18
43	25	79.9434	.0300	3662	75	0747	50	3943	25	1201	17
44	28	78.1263	03	0452	77	20.9460	53	3254	28	0772	16
45	31	76.3900	06	32.7303	80	8188	55	2571	31	0346	15
46	34	74.7292	08	4213	83	6932	58	1893	34	11.9923	14
47	37	73.1390	11	1181	86	5691	61	1222	37	9504	13
48	40	71.6151	14	31.8205	89	4465	64	0557	40	9087	12
49	43	70.1533	17	5284	92	3253	67	14.9898	43	8673	11
50	.0145	68.7501	.0320	31.2416	.0495	20.2056	.0670	14.9244	.0846	11.8262	**10**
51	48	67.4019	23	30.9599	98	0872	73	8596	49	7853	9
52	51	66.1055	26	6833	.0501	19.9702	76	7954	51	7448	8
53	54	64.8580	29	4116	04	8546	79	7317	54	7045	7
54	57	63.6567	32	1446	07	7403	82	6685	57	6645	6
55	60	62.4992	35	29.8823	09	6273	85	6059	60	6248	5
56	63	61.3829	38	6245	12	5156	88	5438	63	5853	4
57	66	60.3058	40	3711	15	4051	90	4823	66	5461	3
58	69	59.2659	43	1220	18	2959	93	4212	69	5072	2
59	72	58.2612	46	28.8771	21	1879	96	3607	72	4685	1
60	.0175	57.2900	.0349	28.6363	.0524	19.0811	.0699	14.3007	.0875	11.4301	**0**
	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	
′	89°		88°		87°		86°		85°		′

′	5°		6°		7°		8°		9°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	**60**
1	745	617	482	449	216	251	946	023	672	764	59
2	774	614	511	446	245	248	975	019	701	760	58
3	803	612	540	443	274	244	.14004	015	730	755	57
4	831	609	569	440	302	240	033	011	758	751	56
5	860	607	597	437	331	237	061	006	787	746	55
6	889	604	626	434	360	233	090	002	816	741	54
7	918	602	655	431	389	230	119	.98998	845	737	53
8	947	599	684	428	418	226	148	994	873	732	52
9	976	596	713	424	447	222	177	990	902	728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	**50**
11	034	591	771	418	504	215	234	982	959	718	49
12	063	588	800	415	533	211	263	978	988	714	48
13	092	586	829	412	562	208	292	973	.16017	709	47
14	121	583	858	409	591	204	320	969	046	704	46
15	150	580	887	406	620	200	349	965	074	700	45
16	179	578	916	402	649	197	378	961	103	695	44
17	208	575	945	399	678	193	407	957	132	690	43
18	237	572	973	396	706	189	436	953	160	686	42
19	266	570	.11002	393	735	186	464	948	189	681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	**40**
21	324	564	060	386	793	178	522	940	246	671	39
22	353	562	089	383	822	175	551	936	275	667	38
23	382	559	118	380	851	171	580	931	304	662	37
24	411	556	147	377	880	167	608	927	333	657	36
25	440	553	176	374	908	163	637	923	361	652	35
26	469	551	205	370	937	160	666	919	390	648	34
27	498	548	234	367	966	156	695	914	419	643	33
28	527	545	263	364	995	152	723	910	447	638	32
29	556	542	291	360	.13024	148	752	906	476	633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	**30**
31	614	537	349	354	081	141	810	897	533	624	29
32	642	534	378	351	110	137	838	893	562	619	28
33	671	531	407	347	139	133	867	889	591	614	27
34	700	528	436	344	168	129	896	884	620	609	26
35	729	526	465	341	197	125	925	880	648	604	25
36	758	523	494	337	226	122	954	876	677	600	24
37	787	520	523	334	254	118	982	871	706	595	23
38	816	517	552	331	283	114	.15011	867	734	590	22
39	845	514	580	327	312	110	040	863	763	585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	**20**
41	903	508	638	320	370	102	097	854	820	575	19
42	932	506	667	317	399	098	126	849	849	570	18
43	961	503	696	314	427	094	155	845	878	565	17
44	990	500	725	310	456	091	184	841	906	561	16
45	.10019	497	754	307	485	087	212	836	935	556	15
46	048	494	783	303	514	083	241	832	964	551	14
47	077	491	812	300	543	079	270	827	992	546	13
48	106	488	840	297	572	075	299	823	.17021	541	12
49	135	485	869	293	600	071	327	818	050	536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	**10**
51	192	479	927	286	658	063	385	809	107	526	9
52	221	476	956	283	687	059	414	805	136	521	8
53	250	473	985	279	716	055	442	800	164	516	7
54	279	470	.12014	276	744	051	471	796	193	511	6
55	308	467	043	272	773	047	500	791	222	506	5
56	337	464	071	269	802	043	529	787	250	501	4
57	366	461	100	265	831	039	557	782	279	496	3
58	395	458	129	262	860	035	586	778	308	491	2
59	424	455	158	258	889	031	615	773	336	486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	**0**
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	
′	84°		83°		82°		81°		80°		′

′	5°		6°		7°		8°		9°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.0875	11.4301	.1051	9.5144	.1228	8.1443	.1405	7.1154	.1584	6.3138	**60**
1	78	11.3919	54	9.4878	31	248	08	004	87	019	59
2	81	540	57	614	34	054	11	7.0855	90	6.2901	58
3	84	163	60	352	37	8.0860	14	706	93	783	57
4	87	11.2789	63	090	40	667	17	558	96	666	56
5	90	417	66	9.3831	43	476	20	410	99	549	55
6	92	048	69	572	46	285	23	264	.1602	432	54
7	95	11.1681	72	315	49	095	26	117	05	316	53
8	98	316	75	060	51	7.9906	29	6.9972	08	200	52
9	.0901	11.0954	78	9.2806	54	718	32	827	11	085	51
10	.0904	11.0594	.1080	9.2553	.1257	7.9530	.1435	6.9682	.1614	6.1970	**50**
11	07	237	83	302	60	344	38	538	17	856	49
12	10	10.9882	86	052	63	158	41	395	20	742	48
13	13	529	89	9.1803	66	7.8973	44	252	23	628	47
14	16	178	92	555	69	789	47	110	26	515	46
15	19	10.8829	95	309	72	606	50	6.8969	29	402	45
16	22	483	98	065	75	424	53	828	32	290	44
17	25	139	.1101	9.0821	78	243	56	687	35	178	43
18	28	10.7797	04	579	81	062	59	548	38	066	42
19	31	457	07	338	84	7.7882	62	408	41	6.0955	41
20	.0934	10.7119	.1110	9.0098	.1287	7.7704	.1465	6.8269	.1644	6.0844	**40**
21	36	10.6783	13	8.9860	90	525	68	131	47	734	39
22	39	450	16	623	93	348	71	6.7994	50	624	38
23	42	118	19	387	96	171	74	856	53	514	37
24	45	10.5789	22	152	99	7.6996	77	720	55	405	36
25	48	462	25	8.8919	.1302	821	80	584	58	296	35
26	51	136	28	686	05	647	83	448	61	188	34
27	54	10.4813	31	455	08	473	86	313	64	080	33
28	57	491	33	225	11	301	89	179	67	5.9972	32
29	60	172	36	8.7996	14	129	92	045	70	865	31
30	.0963	10.3854	.1139	8.7769	.1317	7.5958	.1495	6.6912	.1673	5.9758	**30**
31	66	538	42	542	19	787	97	779	76	651	29
32	69	224	45	317	22	618	.1500	646	79	545	28
33	72	10.2913	48	093	25	449	03	514	82	439	27
34	75	602	51	8.6870	28	281	06	383	85	333	26
35	78	294	54	648	31	113	09	252	88	228	25
36	81	10.1988	57	427	34	7.4947	12	122	91	124	24
37	83	683	60	208	37	781	15	6.5992	94	019	23
38	86	381	63	8.5989	40	615	18	863	97	5.8915	22
39	89	080	66	772	43	451	21	734	.1700	811	21
40	.0992	10.0780	.1169	8.5555	.1346	7.4287	.1524	6.5606	.1703	5.8708	**20**
41	95	483	72	340	49	124	27	478	06	605	19
42	98	187	75	126	52	7.3962	30	350	09	502	18
43	.1001	9.9893	78	8.4913	55	800	33	223	12	400	17
44	04	601	81	701	58	639	36	097	15	298	16
45	07	310	84	490	61	479	39	6.4971	18	197	15
46	10	021	87	280	64	319	42	846	21	095	14
47	13	9.8734	89	071	67	160	45	721	24	5.7994	13
48	16	448	92	8.3863	70	002	48	596	27	894	12
49	19	164	95	656	73	7.2844	51	472	30	794	11
50	.1022	9.7882	.1198	8.3450	.1376	7.2687	.1554	6.4348	.1733	5.7694	**10**
51	25	601	.1201	245	79	531	57	225	36	594	9
52	28	322	04	041	82	375	60	103	39	495	8
53	30	044	07	8.2838	85	220	63	6.3980	42	396	7
54	33	9.6768	10	636	88	066	66	859	45	297	6
55	36	493	13	434	91	7.1912	69	737	48	199	5
56	39	220	16	234	94	759	72	617	51	101	4
57	42	9.5949	19	035	97	607	75	496	54	004	3
58	45	679	22	8.1837	99	455	78	376	57	5.6906	2
59	48	411	25	640	.1402	304	81	257	60	809	1
60	.1051	9.5144	.1228	8.1443	.1405	7.1154	.1584	6.3138	.1763	5.6713	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	84°		83°		82°		81°		80°		′

′	10°		11°		12°		13°		14°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	393	476	109	157	820	809	523	430	220	023	59
2	422	471	138	152	848	803	552	424	249	015	58
3	451	466	167	146	877	797	580	417	277	008	57
4	479	461	195	140	905	791	608	411	305	001	56
5	508	455	224	135	933	784	637	404	333	.96994	55
6	537	450	252	129	962	778	665	398	362	987	54
7	565	445	281	124	990	772	693	391	390	980	53
8	594	440	309	118	.21019	766	722	384	418	973	52
9	623	435	338	112	047	760	750	378	446	966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	680	425	395	101	104	748	807	365	503	952	49
12	708	420	423	096	132	742	835	358	531	945	48
13	737	414	452	090	161	735	863	351	559	937	47
14	766	409	481	084	189	729	892	345	587	930	46
15	794	404	509	079	218	723	920	338	615	923	45
16	823	399	538	073	246	717	948	331	644	916	44
17	852	394	566	067	275	711	977	325	672	909	43
18	880	389	595	061	303	705	.23005	318	700	902	42
19	909	383	623	056	331	698	033	311	728	894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	966	373	680	044	388	686	090	298	784	880	39
22	995	368	709	039	417	680	118	291	813	873	38
23	.18023	362	737	033	445	673	146	284	841	866	37
24	052	357	766	027	474	667	175	278	869	858	36
25	081	352	794	021	502	661	203	271	897	851	35
26	109	347	823	016	530	655	231	264	925	844	34
27	138	341	851	010	559	648	260	257	954	837	33
28	166	336	880	004	587	642	288	251	982	829	32
29	195	331	908	.97998	616	636	316	244	.25010	822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	252	320	965	987	672	623	373	230	066	807	29
32	281	315	994	981	701	617	401	223	094	800	28
33	309	310	.20022	975	729	611	429	217	122	793	27
34	338	304	051	969	758	604	458	210	151	786	26
35	367	299	079	963	786	598	486	203	179	778	25
36	395	294	108	958	814	592	514	196	207	771	24
37	424	288	136	952	843	585	542	189	235	764	23
38	452	283	165	946	871	579	571	182	263	756	22
39	481	277	193	940	899	573	599	176	291	749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	538	267	250	928	956	560	656	162	348	734	19
42	567	261	279	922	985	553	684	155	376	727	18
43	595	256	307	916	.22013	547	712	148	404	719	17
44	624	250	336	910	041	541	740	141	432	712	16
45	652	245	364	905	070	534	769	134	460	705	15
46	681	240	393	899	098	528	797	127	488	697	14
47	710	234	421	893	126	521	825	120	516	690	13
48	738	229	450	887	155	515	853	113	545	682	12
49	767	223	478	881	183	508	882	106	573	675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	824	212	535	869	240	496	938	093	629	660	9
52	852	207	563	863	268	489	966	086	657	653	8
53	881	201	592	857	297	483	995	079	685	645	7
54	910	196	620	851	325	476	.24023	072	713	638	6
55	938	190	649	845	353	470	051	065	741	630	5
56	967	185	677	839	382	463	079	058	769	623	4
57	995	179	706	833	410	457	108	051	798	615	3
58	.19024	174	734	827	438	450	136	044	826	608	2
59	052	168	763	821	467	444	164	037	854	600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	79°		78°		77°		76°		75°		′

′	10°		11°		12°		13°		14°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.1763	5.6713	.1944	5.1446	.2126	4.7046	.2309	4.3315	.2493	4.0108	**60**
1	66	617	47	366	29	4.6979	12	257	96	058	59
2	69	521	50	286	32	912	15	200	99	009	58
3	72	425	53	207	35	845	18	143	.2503	3.9959	57
4	75	329	56	128	38	779	21	086	06	910	56
5	78	234	59	049	41	712	24	029	09	861	55
6	81	140	62	5.0970	44	646	27	4.2972	12	812	54
7	84	045	65	892	47	580	30	916	15	763	53
8	87	5.5951	68	814	50	514	33	859	18	714	52
9	90	857	71	736	53	448	36	803	21	665	51
10	.1793	5.5764	.1974	5.0658	.2156	4.6382	.2339	4.2747	.2524	3.9617	**50**
11	96	671	77	581	59	317	42	691	27	568	49
12	99	578	80	504	62	252	45	635	30	520	48
13	.1802	485	83	427	65	187	49	580	33	471	47
14	05	393	86	350	68	122	52	524	37	423	46
15	08	301	89	273	71	057	55	468	40	375	45
16	11	209	92	197	74	4.5993	58	413	43	327	44
17	14	118	95	121	77	928	61	358	46	279	43
18	17	026	98	045	80	864	64	303	49	232	42
19	20	5.4936	.2001	4.9969	83	800	67	248	52	184	41
20	.1823	5.4845	.2004	4.9894	.2186	4.5736	.2370	4.2193	.2555	3.9136	**40**
21	26	755	07	819	89	673	73	139	58	089	39
22	29	665	10	744	93	609	76	084	61	042	38
23	32	575	13	669	96	546	79	030	64	3.8995	37
24	35	486	16	594	99	483	82	4.1976	68	947	36
25	38	397	19	520	.2202	420	85	922	71	900	35
26	41	308	22	446	05	357	88	868	74	854	34
27	44	219	25	372	08	294	92	814	77	807	33
28	47	131	28	298	11	232	95	760	80	760	32
29	50	043	31	225	14	169	98	706	83	714	31
30	.1853	5.3955	.2035	4.9152	.2217	4.5107	.2401	4.1653	.2586	3.8667	**30**
31	56	868	38	078	20	045	04	600	89	621	29
32	59	781	41	006	23	4.4983	07	547	92	575	28
33	62	694	44	4.8933	26	922	10	493	95	528	27
34	65	607	47	860	29	860	13	441	99	482	26
35	68	521	50	788	32	799	16	388	.2602	436	25
36	71	435	53	716	35	737	19	335	05	391	24
37	74	349	56	644	38	676	22	282	08	345	23
38	77	263	59	573	41	615	25	230	11	299	22
39	80	178	62	501	44	555	28	178	14	254	21
40	.1883	5.3093	.2065	4.8430	.2247	4.4494	.2432	4.1126	.2617	3.8208	**20**
41	87	008	68	359	51	434	35	074	20	163	19
42	90	5.2924	71	288	54	374	38	022	23	118	18
43	93	839	74	218	57	313	41	4.0970	27	073	17
44	96	755	77	147	60	253	44	918	30	028	16
45	99	672	80	077	63	194	47	867	33	3.7983	15
46	.1902	588	83	007	66	134	50	815	36	938	14
47	05	505	86	4.7937	69	075	53	764	39	893	13
48	08	422	89	867	72	015	56	713	42	848	12
49	11	339	92	798	75	4.3956	59	662	45	804	11
50	.1914	5.2257	.2095	4.7729	.2278	4.3897	.2462	4.0611	.2648	3.7760	**10**
51	17	174	98	659	81	838	65	560	51	715	9
52	20	092	.2101	591	84	779	69	509	55	671	8
53	23	011	04	522	87	721	72	459	58	627	7
54	26	5.1929	07	453	90	662	75	408	61	583	6
55	29	848	10	385	93	604	78	358	64	539	5
56	32	767	13	317	96	546	81	308	67	495	4
57	35	686	16	249	99	488	84	257	70	451	3
58	38	606	19	181	.2303	430	87	207	73	408	2
59	41	526	23	114	06	372	90	158	76	364	1
60	.1944	5.1446	.2126	4.7046	.2309	4.3315	.2493	4.0108	.2679	3.7321	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	**60**
1	910	585	592	118	265	622	929	097	584	542	59
2	938	578	620	110	293	613	957	088	612	533	58
3	966	570	648	102	321	605	985	079	639	523	57
4	994	562	676	094	348	596	.31012	070	667	514	56
5	.26022	555	704	086	376	588	040	061	694	504	55
6	050	547	731	078	404	579	068	052	722	495	54
7	079	540	759	070	432	571	095	043	749	485	53
8	107	532	787	062	460	562	123	033	777	476	52
9	135	524	815	054	487	554	151	024	804	466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	**50**
11	191	509	871	037	543	536	206	006	859	447	49
12	219	502	899	029	571	528	233	.94997	887	438	48
13	247	494	927	021	599	519	261	988	914	428	47
14	275	486	955	013	626	511	289	979	942	418	46
15	303	479	983	005	654	502	316	970	969	409	45
16	331	471	.28011	.95997	682	493	344	961	997	399	44
17	359	463	039	989	710	485	372	952	.33024	390	43
18	387	456	067	981	737	476	399	943	051	380	42
19	415	448	095	972	765	467	427	933	079	370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	**40**
21	471	433	150	956	821	450	482	915	134	351	39
22	500	425	178	948	849	441	510	906	161	342	38
23	528	417	206	940	876	433	537	897	189	332	37
24	556	410	234	931	904	424	565	888	216	322	36
25	584	402	262	923	932	415	593	878	244	313	35
26	612	394	290	915	960	407	620	869	271	303	34
27	640	386	318	907	987	398	648	860	298	293	33
28	668	379	346	898	.30015	389	675	851	326	284	32
29	696	371	374	890	043	380	703	842	353	274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	**30**
31	752	355	429	874	098	363	758	823	408	254	29
32	780	347	457	865	126	354	786	814	436	245	28
33	808	340	485	857	154	345	813	805	463	235	27
34	836	332	513	849	182	337	841	795	490	225	26
35	864	324	541	841	209	328	868	786	518	215	25
36	892	316	569	832	237	319	896	777	545	206	24
37	920	308	597	824	265	310	923	768	573	196	23
38	948	301	625	816	292	301	951	758	600	186	22
39	976	293	652	807	320	293	979	749	627	176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	**20**
41	032	277	708	791	376	275	034	730	682	157	19
42	060	269	736	782	403	266	061	721	710	147	18
43	088	261	764	774	431	257	089	712	737	137	17
44	116	253	792	766	459	248	116	702	764	127	16
45	144	246	820	757	486	240	144	693	792	118	15
46	172	238	847	749	514	231	171	684	819	108	14
47	200	230	875	740	542	222	199	674	846	098	13
48	228	222	903	732	570	213	227	665	874	088	12
49	256	214	931	724	597	204	254	656	901	078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	**10**
51	312	198	987	707	653	186	309	637	956	058	9
52	340	190	.29015	698	680	177	337	627	983	049	8
53	368	182	042	690	708	168	364	618	.34011	039	7
54	396	174	070	681	736	159	392	609	038	029	6
55	424	166	098	673	763	150	419	599	065	019	5
56	452	158	126	664	791	142	447	590	093	009	4
57	480	150	154	656	819	133	474	580	120	.93999	3
58	508	142	182	647	846	124	502	571	147	989	2
59	536	134	209	639	874	115	529	561	175	979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	74°		73°		72°		71°		70°		′

′	15°		16°		17°		18°		19°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.2679	3.7321	.2867	3.4874	.3057	3.2709	.3249	3.0777	.3443	2.9042	**60**
1	83	277	71	836	60	675	52	746	47	015	59
2	86	234	74	798	64	641	56	716	50	2.8987	58
3	89	191	77	760	67	607	59	686	53	960	57
4	92	148	80	722	70	573	62	655	56	933	56
5	95	105	83	684	73	539	65	625	60	905	55
6	98	062	86	646	76	506	69	595	63	878	54
7	.2701	019	90	608	80	472	72	565	66	851	53
8	04	3.6976	93	570	83	438	75	535	69	824	52
9	08	933	96	533	86	405	78	505	73	797	51
10	.2711	3.6891	.2899	3.4495	.3089	3.2371	.3281	3.0475	.3476	2.8770	**50**
11	14	848	.2902	458	92	338	85	445	79	743	49
12	17	806	05	420	96	305	88	415	82	716	48
13	20	764	08	383	99	272	91	385	86	689	47
14	23	722	12	346	.3102	238	94	356	89	662	46
15	26	680	15	308	05	205	98	326	92	636	45
16	29	638	18	271	08	172	.3301	296	95	609	44
17	33	596	21	234	11	139	04	267	99	582	43
18	36	554	24	197	15	106	07	237	.3502	556	42
19	39	512	27	160	18	073	10	208	05	529	41
20	.2742	3.6470	.2931	3.4124	.3121	3.2041	.3314	3.0178	.3508	2.8502	**40**
21	45	429	34	087	24	008	17	149	12	476	39
22	48	387	37	050	27	3.1975	20	120	15	449	38
23	51	346	40	014	31	943	23	090	18	423	37
24	54	305	43	3.3977	34	910	27	061	22	397	36
25	58	264	46	941	37	878	30	032	25	370	35
26	61	222	49	904	40	845	33	003	28	344	34
27	64	181	53	868	43	813	36	2.9974	31	318	33
28	67	140	56	832	47	780	39	945	35	291	32
29	70	100	59	796	50	748	43	916	38	265	31
30	.2773	3.6059	.2962	3.3759	.3153	3.1716	.3346	2.9887	.3541	2.8239	**30**
31	76	018	65	723	56	684	49	858	44	213	29
32	80	3.5978	68	687	59	652	52	829	48	187	28
33	83	937	72	652	63	620	56	800	51	161	27
34	86	897	75	616	66	588	59	772	54	135	26
35	89	856	78	580	69	556	62	743	58	109	25
36	92	816	81	544	72	524	65	714	61	083	24
37	95	776	84	509	75	492	69	686	64	057	23
38	98	736	87	473	79	460	72	657	67	032	22
39	.2801	696	91	438	82	429	75	629	71	006	21
40	.2805	3.5656	.2994	3.3402	.3185	3.1397	.3378	2.9600	.3574	2.7980	**20**
41	08	616	97	367	88	366	82	572	77	955	19
42	11	576	.3000	332	91	334	85	544	81	929	18
43	14	536	03	297	95	303	88	515	84	903	17
44	17	497	06	261	98	271	91	487	87	878	16
45	20	457	10	226	.3201	240	95	459	90	852	15
46	23	418	13	191	04	209	98	431	94	827	14
47	27	379	16	156	07	178	.3401	403	97	801	13
48	30	339	19	122	11	146	04	375	.3600	776	12
49	33	300	22	087	14	115	08	347	04	751	11
50	.2836	3.5261	.3026	3.3052	.3217	3.1084	.3411	2.9319	.3607	2.7725	**10**
51	39	222	29	017	20	053	14	291	10	700	9
52	42	183	32	3.2983	23	022	17	263	13	675	8
53	45	144	35	948	27	3.0991	21	235	17	650	7
54	49	105	38	914	30	961	24	208	20	625	6
55	52	067	41	879	33	930	27	180	23	600	5
56	55	028	45	845	36	899	30	152	27	575	4
57	58	3.4989	48	811	40	868	34	125	30	550	3
58	61	951	51	777	43	838	37	097	33	525	2
59	64	912	54	743	46	807	40	070	36	500	1
60	.2867	3.4874	.3057	3.2709	.3249	3.0777	.3443	2.9042	.3640	2.7475	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	229	959	864	348	488	707	100	039	700	343	59
2	257	949	891	337	515	697	127	028	727	331	58
3	284	939	918	327	542	686	153	016	753	319	57
4	311	929	945	316	569	675	180	005	780	307	56
5	339	919	973	306	595	664	207	.91994	806	295	55
6	366	909	.36000	295	622	653	234	982	833	283	54
7	393	899	027	285	649	642	260	971	860	272	53
8	421	889	054	274	676	631	287	959	886	260	52
9	448	879	081	264	703	620	314	948	913	248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	503	859	135	243	757	598	367	925	966	224	49
12	530	849	162	232	784	587	394	914	992	212	48
13	557	839	190	222	811	576	421	902	.41019	200	47
14	584	829	217	211	838	565	448	891	045	188	46
15	612	819	244	201	865	554	474	879	072	176	45
16	639	809	271	190	892	543	501	868	098	164	44
17	666	799	298	180	919	532	528	856	125	152	43
18	694	789	325	169	946	521	555	845	151	140	42
19	721	779	352	159	973	510	581	833	178	128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	775	759	406	137	.38026	488	635	810	231	104	39
22	803	748	434	127	053	477	661	799	257	092	38
23	830	738	461	116	080	466	688	787	284	080	37
24	857	728	488	106	107	455	715	775	310	068	36
25	884	718	515	095	134	444	741	764	337	056	35
26	912	708	542	084	161	432	768	752	363	044	34
27	939	698	569	074	188	421	795	741	390	032	33
28	966	688	596	063	215	410	822	729	416	020	32
29	993	677	623	052	241	399	848	718	443	008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	048	657	677	031	295	377	902	694	496	984	29
32	075	647	704	020	322	366	928	683	522	972	28
33	102	637	731	010	349	355	955	671	549	960	27
34	130	626	758	.92999	376	343	982	660	575	948	26
35	157	616	785	988	403	332	.40008	648	602	936	25
36	184	606	812	978	430	321	035	636	628	924	24
37	211	596	839	967	456	310	062	625	655	911	23
38	239	585	867	956	483	299	088	613	681	899	22
39	266	575	894	945	510	287	115	601	707	887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	320	555	948	924	564	265	168	578	760	863	19
42	347	544	975	913	591	254	195	566	787	851	18
43	375	534	.37002	902	617	243	221	555	813	839	17
44	402	524	029	892	644	231	248	543	840	826	16
45	429	514	056	881	671	220	275	531	866	814	15
46	456	503	083	870	698	209	301	519	892	802	14
47	484	493	110	859	725	198	328	508	919	790	13
48	511	483	137	849	752	186	355	496	945	778	12
49	538	472	164	838	778	175	381	484	972	766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	592	452	218	816	832	152	434	461	.42024	741	9
52	619	441	245	805	859	141	461	449	051	729	8
53	647	431	272	794	886	130	488	437	077	717	7
54	674	420	299	784	912	119	514	425	104	704	6
55	701	410	326	773	939	107	541	414	130	692	5
56	728	400	353	762	966	096	567	402	156	680	4
57	755	389	380	751	993	085	594	390	183	668	3
58	782	379	407	740	.39020	073	621	378	209	655	2
59	810	368	434	729	046	062	647	366	235	643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	69°		68°		67°		66°		65°		′

′	20°		21°		22°		23°		24°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.3640	2.7475	.3839	2.6051	.4040	2.4751	.4245	2.3559	.4452	2.2460	**60**
1	43	50	42	28	44	30	48	39	56	43	59
2	46	25	45	06	47	09	52	20	59	25	58
3	50	00	49	2.5983	50	2.4689	55	01	63	08	57
4	53	2.7376	52	61	54	68	58	2.3483	66	2.2390	56
5	56	51	55	38	57	48	62	64	70	73	55
6	59	26	59	16	61	27	65	45	73	55	54
7	63	02	62	2.5893	64	06	69	26	77	38	53
8	66	2.7277	65	71	67	2.4586	72	07	80	20	52
9	69	53	69	48	71	66	76	2.3388	84	03	51
10	.3673	2.7228	.3872	2.5826	.4074	2.4545	.4279	2.3369	.4487	2.2286	**50**
11	76	04	75	04	78	25	83	51	91	68	49
12	79	2.7179	79	2.5782	81	04	86	32	94	51	48
13	83	55	82	59	84	2.4484	89	13	98	34	47
14	86	30	85	37	88	64	93	2.3294	.4501	16	46
15	89	06	89	15	91	43	96	76	05	2.2199	45
16	92	2.7082	92	2.5693	95	23	.4300	57	08	82	44
17	96	58	95	71	98	03	03	38	12	65	43
18	99	34	99	49	.4101	2.4383	07	20	15	48	42
19	.3702	09	.3902	27	05	62	10	01	19	30	41
20	.3706	2.6985	.3906	2.5605	.4108	2.4342	.4314	2.3183	.4522	2.2113	**40**
21	09	61	09	2.5583	11	22	17	64	26	2.2096	39
22	12	37	12	61	15	02	20	46	29	79	38
23	16	13	16	39	18	2.4282	24	27	33	62	37
24	19	2.6889	19	17	22	62	27	09	36	45	36
25	22	65	22	2.5495	25	42	31	2.3090	40	28	35
26	26	41	26	73	29	22	34	72	43	11	34
27	29	18	29	52	32	02	38	53	47	2.1994	33
28	32	2.6794	32	30	35	2.4182	41	35	50	77	32
29	36	70	36	08	39	62	45	17	54	60	31
30	.3739	2.6746	.3939	2.5386	.4142	2.4142	.4348	2.2998	.4557	2.1943	**30**
31	42	23	42	65	46	22	52	80	61	26	29
32	45	2.6699	46	43	49	02	55	62	64	09	28
33	49	75	49	22	52	2.4083	59	44	68	2.1892	27
34	52	52	53	00	56	63	62	25	71	76	26
35	55	28	56	2.5279	59	43	65	07	75	59	25
36	59	05	59	57	63	23	69	2.2889	78	42	24
37	62	2.6581	63	36	66	04	72	71	82	25	23
38	65	58	66	14	69	2.3984	76	53	85	08	22
39	69	34	69	2.5193	73	64	79	35	89	2.1792	21
40	.3772	2.6511	.3973	2.5172	.4176	2.3945	.4383	2.2817	.4592	2.1775	**20**
41	75	2.6488	76	50	80	25	86	2.2799	96	58	19
42	79	64	79	29	83	06	90	81	99	42	18
43	82	41	83	08	87	2.3886	93	63	.4603	25	17
44	85	18	86	2.5086	90	67	97	45	07	08	16
45	89	2.6395	90	65	93	47	.4400	27	10	2.1692	15
46	92	71	93	44	97	28	04	09	14	75	14
47	95	48	96	23	.4200	08	07	2.2691	17	59	13
48	99	25	.4000	02	04	2.3789	11	73	21	42	12
49	.3802	02	03	2.4981	07	70	14	55	24	25	11
50	.3805	2.6279	.4006	2.4960	.4210	2.3750	.4417	2.2637	.4628	2.1609	**10**
51	09	56	10	39	14	31	21	20	31	2.1592	9
52	12	33	13	18	17	12	24	02	35	76	8
53	15	10	17	2.4897	21	2.3693	28	2.2584	38	60	7
54	19	2.6187	20	76	24	73	31	66	42	43	6
55	22	65	23	55	28	54	35	49	45	27	5
56	25	42	27	34	31	35	38	31	49	10	4
57	29	19	30	13	34	16	42	13	52	2.1494	3
58	32	2.6096	33	2.4792	38	2.3597	45	2.2496	56	78	2
59	35	74	37	72	41	78	49	78	60	61	1
60	.3839	2.6051	.4040	2.4751	.4245	2.3559	.4452	2.2460	.4663	2.1445	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	69°		68°		67°		66°		65°		′

′	25°		26°		27°		28°		29°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	288	618	863	867	425	087	973	281	506	448	59
2	315	606	889	854	451	074	999	267	532	434	58
3	341	594	916	841	477	061	.47024	254	557	420	57
4	367	582	942	828	503	048	050	240	583	406	56
5	394	569	968	816	529	035	076	226	608	391	55
6	420	557	994	803	554	021	101	213	634	377	54
7	446	545	.44020	790	580	008	127	199	659	363	53
8	473	532	046	777	606	.88995	153	185	684	349	52
9	499	520	072	764	632	981	178	172	710	335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	552	495	124	739	684	955	229	144	761	306	49
12	578	483	151	726	710	942	255	130	786	292	48
13	604	470	177	713	736	928	281	117	811	278	47
14	631	458	203	700	762	915	306	103	837	264	46
15	657	446	229	687	787	902	332	089	862	250	45
16	683	433	255	674	813	888	358	075	888	235	44
17	709	421	281	662	839	875	383	062	913	221	43
18	736	408	307	649	865	862	409	048	938	207	42
19	762	396	333	636	891	848	434	034	964	193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	815	371	385	610	942	822	486	006	.49014	164	39
22	841	358	411	597	968	808	511	.87993	040	150	38
23	867	346	437	584	994	795	537	979	065	136	37
24	894	334	464	571	.46020	782	562	965	090	121	36
25	920	321	490	558	046	768	588	951	116	107	35
26	946	309	516	545	072	755	614	937	141	093	34
27	972	296	542	532	097	741	639	923	166	079	33
28	999	284	568	519	123	728	665	909	192	064	32
29	.43025	271	594	506	149	715	690	896	217	050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	077	246	646	480	201	688	741	868	268	021	29
32	104	233	672	467	226	674	767	854	293	007	28
33	130	221	698	454	252	661	793	840	318	.86993	27
34	156	208	724	441	278	647	818	826	344	978	26
35	182	196	750	428	304	634	844	812	369	964	25
36	209	183	776	415	330	620	869	798	394	949	24
37	235	171	802	402	355	607	895	784	419	935	23
38	261	158	828	389	381	593	920	770	445	921	22
39	287	146	854	376	407	580	946	756	470	906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	340	120	906	350	458	553	997	729	521	878	19
42	366	108	932	337	484	539	.48022	715	546	863	18
43	392	095	958	324	510	526	048	701	571	849	17
44	418	082	984	311	536	512	073	687	596	834	16
45	445	070	.45010	298	561	499	099	673	622	820	15
46	471	057	036	285	587	485	124	659	647	805	14
47	497	045	062	272	613	472	150	645	672	791	13
48	523	032	088	259	639	458	175	631	697	777	12
49	549	019	114	245	664	445	201	617	723	762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	602	.89994	166	219	716	417	252	589	773	733	9
52	628	981	192	206	742	404	277	575	798	719	8
53	654	968	218	193	767	390	303	561	824	704	7
54	680	956	243	180	793	377	328	546	849	690	6
55	706	943	269	167	819	363	354	532	874	675	5
56	733	930	295	153	844	349	379	518	899	661	4
57	759	918	321	140	870	336	405	504	924	646	3
58	785	905	347	127	896	322	430	490	950	632	2
59	811	892	373	114	921	308	456	476	975	617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	64°		63°		62°		61°		60°		′

′	25°		26°		27°		28°		29°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.4663	2.1445	.4877	2.0503	.5095	1.9626	.5317	1.8807	.5543	1.8040	**60**
1	67	29	81	2.0488	99	12	21	1.8794	47	28	59
2	70	13	85	73	.5103	1.9598	25	81	51	16	58
3	74	2.1396	88	58	06	84	28	68	55	03	57
4	77	80	92	43	10	70	32	55	58	1.7991	56
5	81	64	95	28	14	56	36	41	62	79	55
6	84	48	99	13	17	42	40	28	66	66	54
7	88	32	.4903	2.0398	21	28	43	15	70	54	53
8	91	15	06	83	25	14	47	02	74	42	52
9	95	2.1299	10	68	28	00	51	1.8689	77	30	51
10	.4699	2.1283	.4913	2.0353	.5132	1.9486	.5354	1.8676	.5581	1.7917	**50**
11	.4702	67	17	38	36	72	58	63	85	05	49
12	06	51	21	23	39	58	62	50	89	1.7893	48
13	09	35	24	08	43	44	66	37	93	81	47
14	13	19	28	2.0293	47	30	69	24	96	68	46
15	16	03	31	78	50	16	73	11	.5600	56	45
16	20	2.1187	35	63	54	02	77	1.8598	04	44	44
17	23	71	39	48	58	1.9388	81	85	08	32	43
18	27	55	42	33	61	75	84	72	12	20	42
19	31	39	46	19	65	61	88	59	16	08	41
20	.4734	2.1123	.4950	2.0204	.5169	1.9347	.5392	1.8546	.5619	1.7796	**40**
21	38	07	53	2.0189	72	33	96	33	23	83	39
22	41	2.1092	57	74	76	19	99	20	27	71	38
23	45	76	60	60	80	06	.5403	07	31	59	37
24	48	60	64	45	84	1.9292	07	1.8495	35	47	36
25	52	44	68	30	87	78	11	82	39	35	35
26	55	28	71	15	91	65	15	69	42	23	34
27	59	13	75	01	95	51	18	56	46	11	33
28	63	2.0997	79	2.0086	98	37	22	43	50	1.7699	32
29	66	81	82	72	.5202	23	26	30	54	87	31
30	.4770	2.0965	.4986	2.0057	.5206	1.9210	.5430	1.8418	.5658	1.7675	**30**
31	73	50	89	42	09	1.9196	33	05	62	63	29
32	77	34	93	28	13	83	37	1.8392	65	51	28
33	80	18	97	13	17	69	41	79	69	39	27
34	84	03	.5000	1.9999	20	55	45	67	73	27	26
35	88	2.0887	04	84	24	42	48	54	77	15	25
36	91	72	08	70	28	28	52	41	81	03	24
37	95	56	11	55	32	15	56	29	85	1.7591	23
38	98	40	15	41	35	01	60	16	88	79	22
39	.4802	25	19	26	39	1.9088	64	03	92	67	21
40	.4806	2.0809	.5022	1.9912	.5243	1.9074	.5467	1.8291	.5696	1.7556	**20**
41	09	2.0794	26	1.9897	46	61	71	78	.5700	44	19
42	13	78	29	83	50	47	75	65	04	32	18
43	16	63	33	68	54	34	79	53	08	20	17
44	20	48	37	54	58	20	82	40	12	08	16
45	23	32	40	40	61	07	86	28	15	1.7496	15
46	27	17	44	25	65	1.8993	90	15	19	85	14
47	31	01	48	11	69	80	94	02	23	73	13
48	34	2.0686	51	1.9797	72	67	98	1.8190	27	61	12
49	38	71	55	82	76	53	.5501	77	31	49	11
50	.4841	2.0655	.5059	1.9768	.5280	1.8940	.5505	1.8165	.5735	1.7437	**10**
51	45	40	62	54	84	27	09	52	39	26	9
52	49	25	66	40	87	13	13	40	43	14	8
53	52	09	70	25	91	00	17	27	46	02	7
54	56	2.0594	73	11	95	1.8887	20	15	50	1.7391	6
55	59	79	77	1.9697	98	73	24	03	54	79	5
56	63	64	81	83	.5302	60	28	1.8090	58	67	4
57	67	49	84	69	06	47	32	78	62	55	3
58	70	33	88	54	10	34	35	65	66	44	2
59	74	18	92	40	13	20	39	53	70	32	1
60	.4877	2.0503	.5095	1.9626	.5317	1.8807	.5543	1.8040	.5774	1.7321	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	**60**
1	025	588	529	702	.53017	789	488	851	943	887	59
2	050	573	554	687	041	774	513	835	968	871	58
3	076	559	579	672	066	759	537	819	992	855	57
4	101	544	604	657	091	743	561	804	.56016	839	56
5	126	530	628	642	115	728	586	788	040	822	55
6	151	515	653	627	140	712	610	772	064	806	54
7	176	501	678	612	164	697	635	756	088	790	53
8	201	486	703	597	189	681	659	740	112	773	52
9	227	471	728	582	214	666	683	724	136	757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	**50**
11	277	442	778	551	263	635	732	692	184	724	49
12	302	427	803	536	288	619	756	676	208	708	48
13	327	413	828	521	312	604	781	660	232	692	47
14	352	398	852	506	337	588	805	645	256	675	46
15	377	384	877	491	361	573	829	629	280	659	45
16	403	369	902	476	386	557	854	613	305	643	44
17	428	354	927	461	411	542	878	597	329	626	43
18	453	340	952	446	435	526	902	581	353	610	42
19	478	325	977	431	460	511	927	565	377	593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	**40**
21	528	295	026	401	509	480	975	533	425	561	39
22	553	281	051	385	534	464	999	517	449	544	38
23	578	266	076	370	558	448	.55024	501	473	528	37
24	603	251	101	355	583	433	048	485	497	511	36
25	628	237	126	340	607	417	072	469	521	495	35
26	654	222	151	325	632	402	097	453	545	478	34
27	679	207	175	310	656	386	121	437	569	462	33
28	704	192	200	294	681	370	145	421	593	446	32
29	729	178	225	279	705	355	169	405	617	429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	**30**
31	779	148	275	249	754	324	218	373	665	396	29
32	804	133	299	234	779	308	242	356	689	380	28
33	829	119	324	218	804	292	266	340	713	363	27
34	854	104	349	203	828	277	291	324	736	347	26
35	879	089	374	188	853	261	315	308	760	330	25
36	904	074	399	173	877	245	339	292	784	314	24
37	929	059	423	157	902	230	363	276	808	297	23
38	954	045	448	142	926	214	388	260	832	281	22
39	979	030	473	127	951	198	412	244	856	264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	**20**
41	029	000	522	096	.54000	167	460	212	904	231	19
42	054	.85985	547	081	024	151	484	195	928	214	18
43	079	970	572	066	049	135	509	179	952	198	17
44	104	956	597	051	073	120	533	163	976	181	16
45	129	941	621	035	097	104	557	147	.57000	165	15
46	154	926	646	020	122	088	581	131	024	148	14
47	179	911	671	005	146	072	605	115	047	132	13
48	204	896	696	.84989	171	057	630	098	071	115	12
49	229	881	720	974	195	041	654	082	095	098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	**10**
51	279	851	770	943	244	009	702	050	143	065	9
52	304	836	794	928	269	.83994	726	034	167	048	8
53	329	821	819	913	293	978	750	017	191	032	7
54	354	806	844	897	317	962	775	001	215	015	6
55	379	792	869	882	342	946	799	.82985	238	.81999	5
56	404	777	893	866	366	930	823	969	262	982	4
57	429	762	918	851	391	915	847	953	286	965	3
58	454	747	943	836	415	899	871	936	310	949	2
59	479	732	967	820	440	883	895	920	334	932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	**0**
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	
′	59°		58°		57°		56°		55°		′

′	30°		31°		32°		33°		34°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.5774	1.7321	.6009	1.6643	.6249	1.6003	.6494	1.5399	.6745	1.4826	**60**
1	77	09	13	32	53	1.5993	98	89	49	16	59
2	81	1.7297	17	21	57	83	.6502	79	54	07	58
3	85	86	20	10	61	72	06	69	58	1.4798	57
4	89	74	24	1.6599	65	62	11	59	62	88	56
5	93	62	28	88	69	52	15	50	66	79	55
6	97	51	32	77	73	41	19	40	71	70	54
7	.5801	39	36	66	77	31	23	30	75	61	53
8	05	28	40	55	81	21	27	20	79	51	52
9	08	16	44	45	85	11	31	11	83	42	51
10	.5812	1.7205	.6048	1.6534	.6289	1.5900	.6536	1.5301	.6787	1.4733	**50**
11	16	1.7193	52	23	93	1.5890	40	1.5291	92	24	49
12	20	82	56	12	97	80	44	82	96	15	48
13	24	70	60	01	.6301	69	48	72	.6800	05	47
14	28	59	64	1.6490	05	59	52	62	05	1.4696	46
15	32	47	68	79	10	49	56	53	09	87	45
16	36	36	72	69	14	39	60	43	13	78	44
17	40	24	76	58	18	29	65	33	17	69	43
18	44	13	80	47	22	18	69	24	22	1.4659	42
19	47	02	84	36	26	08	73	14	26	50	41
20	.5851	1.7090	.6088	1.6426	.6330	1.5798	.6577	1.5204	.6830	1.4641	**40**
21	55	79	92	15	34	88	81	1.5195	34	32	39
22	59	67	96	04	38	78	85	85	39	23	38
23	63	56	.6100	1.6393	42	68	90	75	43	14	37
24	67	45	04	83	46	57	94	66	47	05	36
25	71	33	08	72	50	47	98	56	51	1.4596	35
26	75	22	12	61	54	37	.6602	47	56	86	34
27	79	11	16	51	58	27	06	37	60	77	33
28	83	1.6999	20	40	63	17	10	27	64	68	32
29	87	88	24	29	67	07	15	18	69	59	31
30	.5890	1.6977	.6128	1.6319	.6371	1.5697	.6619	1.5108	.6873	1.4550	**30**
31	94	65	32	08	75	87	23	1.5099	77	41	29
32	98	54	36	1.6297	79	77	27	89	81	32	28
33	.5902	43	40	87	83	67	31	80	86	23	27
34	06	32	44	76	87	57	36	70	90	14	26
35	10	20	48	65	91	47	40	61	94	05	25
36	14	09	52	55	95	37	44	51	99	1.4496	24
37	18	1.6898	56	44	99	27	48	42	.6903	87	23
38	22	87	60	34	.6403	17	52	32	07	78	22
39	26	75	64	23	08	07	57	23	11	69	21
40	.5930	1.6864	.6168	1.6212	.6412	1.5597	.6661	1.5013	.6916	1.4460	**20**
41	34	53	72	02	16	87	65	04	20	51	19
42	38	42	76	1.6191	20	77	69	1.4994	24	42	18
43	42	31	80	81	24	67	73	85	29	33	17
44	45	20	84	70	28	57	78	75	33	24	16
45	49	08	88	60	32	47	82	66	37	15	15
46	53	1.6797	92	49	36	37	86	57	42	06	14
47	57	86	96	39	40	27	90	47	46	1.4397	13
48	61	75	.6200	28	45	17	94	38	50	88	12
49	65	64	04	18	49	07	99	28	54	79	11
50	.5969	1.6753	.6208	1.6107	.6453	1.5497	.6703	1.4919	.6959	1.4370	**10**
51	73	42	12	1.6097	57	87	07	10	63	61	9
52	77	31	16	87	61	77	11	00	67	52	8
53	81	20	20	76	65	68	15	1.4891	72	44	7
54	85	09	24	66	69	58	20	82	76	35	6
55	89	1.6698	28	55	73	48	24	72	80	26	5
56	93	87	33	45	78	38	28	63	85	17	4
57	97	76	37	34	82	28	32	54	89	08	3
58	.6001	65	41	24	86	18	37	44	93	1.4299	2
59	05	54	45	14	90	08	41	35	98	90	1
60	.6009	1.6643	.6249	1.6003	.6494	1.5399	.6745	1.4826	.7002	1.4281	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	**60**
1	381	899	802	885	205	846	589	783	955	696	59
2	405	882	826	867	228	829	612	765	977	678	58
3	429	865	849	850	251	811	635	747	.63000	660	57
4	453	848	873	833	274	793	658	729	022	641	56
5	477	832	896	816	298	776	681	711	045	623	55
6	501	815	920	799	321	758	704	694	068	605	54
7	524	798	943	782	344	741	726	676	090	586	53
8	548	782	967	765	367	723	749	658	113	568	52
9	572	765	990	748	390	706	772	640	135	550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	**50**
11	619	731	037	713	437	671	818	604	180	513	49
12	643	714	061	696	460	653	841	586	203	494	48
13	667	698	084	679	483	635	864	568	225	476	47
14	691	681	108	662	506	618	887	550	248	458	46
15	715	664	131	644	529	600	909	532	271	439	45
16	738	647	154	627	553	583	932	514	293	421	44
17	762	631	178	610	576	565	955	496	316	402	43
18	786	614	201	593	599	547	978	478	338	384	42
19	810	597	225	576	622	530	.62001	460	361	366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	**40**
21	857	563	272	541	668	494	046	424	406	329	39
22	881	546	295	524	691	477	069	405	428	310	38
23	904	530	318	507	714	459	092	387	451	292	37
24	928	513	342	489	738	441	115	369	473	273	36
25	952	496	365	472	761	424	138	351	496	255	35
26	976	479	389	455	784	406	160	333	518	236	34
27	999	462	412	438	807	388	183	315	540	218	33
28	.58023	445	436	420	830	371	206	297	563	199	32
29	047	428	459	403	853	353	229	279	585	181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	**30**
31	094	395	506	368	899	318	274	243	630	144	29
32	118	378	529	351	922	300	297	225	653	125	28
33	141	361	552	334	945	282	320	206	675	107	27
34	165	344	576	316	968	264	342	188	698	088	26
35	189	327	599	299	991	247	365	170	720	070	25
36	212	310	622	282	.61015	229	388	152	742	051	24
37	236	293	646	264	038	211	411	134	765	033	23
38	260	276	669	247	061	193	433	116	787	014	22
39	283	259	693	230	084	176	456	098	810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	**20**
41	330	225	739	195	130	140	502	061	854	959	19
42	354	208	763	178	153	122	524	043	877	940	18
43	378	191	786	160	176	105	547	025	899	921	17
44	401	174	809	143	199	087	570	007	922	903	16
45	425	157	832	125	222	069	592	.77988	944	884	15
46	449	140	856	108	245	051	615	970	966	866	14
47	472	123	879	091	268	033	638	952	989	847	13
48	496	106	902	073	291	016	660	934	.64011	828	12
49	519	089	926	056	314	.78998	683	916	033	810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	**10**
51	567	055	972	021	360	962	728	879	078	772	9
52	590	038	995	003	383	944	751	861	100	754	8
53	614	021	.60019	.79986	406	926	774	843	123	735	7
54	637	004	042	968	429	908	796	824	145	717	6
55	661	.80987	065	951	451	891	819	806	167	698	5
56	684	970	089	934	474	873	842	788	190	679	4
57	708	953	112	916	497	855	864	769	212	661	3
58	731	936	135	899	520	837	887	751	234	642	2
59	755	919	158	881	543	819	909	733	256	623	1
60	.58779	.80902	.60182	.79864	.61566	78801	.62932	.77715	.64279	.76604	**0**
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	
′	54°		53°		52°		51°		50°		′

′	35°		36°		37°		38°		39°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.7002	1.4281	.7265	1.3764	.7536	1.3270	.7813	1.2799	.8098	1.2349	**60**
1	06	73	70	55	40	62	18	92	.8103	42	59
2	11	64	74	47	45	54	22	84	07	34	58
3	15	55	79	39	49	46	27	76	12	27	57
4	19	46	83	30	54	38	32	69	17	20	56
5	24	37	88	22	58	30	36	61	22	12	55
6	28	29	92	13	63	22	41	53	27	05	54
7	32	20	97	05	68	14	46	46	32	1.2298	53
8	37	11	.7301	1.3697	72	06	50	38	36	90	52
9	41	02	06	88	77	1.3198	55	31	41	83	51
10	.7046	1.4193	.7310	1.3680	.7581	1.3190	.7860	1.2723	.8146	1.2276	**50**
11	50	85	14	72	86	82	65	15	51	68	49
12	54	76	19	63	90	75	69	08	56	61	48
13	59	67	23	55	95	67	74	00	61	54	47
14	63	58	28	47	.7600	59	79	1.2693	65	47	46
15	67	50	32	38	04	51	83	85	70	39	45
16	72	41	37	30	09	43	88	77	75	32	44
17	76	32	41	22	13	35	93	70	80	25	43
18	80	24	46	13	18	27	98	62	85	18	42
19	85	15	50	05	23	19	.7902	55	90	10	41
20	.7089	1.4106	.7355	1.3597	.7627	1.3111	.7907	1.2647	.8195	1.2203	**40**
21	94	1.4097	59	88	32	03	12	40	99	1.2196	39
22	98	89	64	80	36	1.3095	16	32	.8204	89	38
23	.7102	80	68	72	41	87	21	24	09	81	37
24	07	71	73	64	46	79	26	17	14	74	36
25	11	63	77	55	50	72	31	09	19	67	35
26	15	54	82	47	55	64	35	02	24	60	34
27	20	45	86	39	59	56	40	1.2594	29	53	33
28	24	37	91	31	64	48	45	87	34	45	32
29	29	28	95	22	69	40	50	79	38	38	31
30	.7133	1.4019	.7400	1.3514	.7673	1.3032	.7954	1.2572	.8243	1.2131	**30**
31	37	11	04	06	78	24	59	64	48	24	29
32	42	02	09	1.3498	83	17	64	57	53	17	28
33	46	1.3994	13	90	87	09	69	49	58	09	27
34	51	85	18	81	92	01	73	42	63	02	26
35	55	76	22	73	96	1.2993	78	34	68	1.2095	25
36	59	68	27	65	.7701	85	83	27	73	88	24
37	64	59	31	57	06	77	88	19	78	81	23
38	68	51	36	49	10	70	92	12	83	74	22
39	73	42	40	40	15	62	97	04	87	66	21
40	.7177	1.3934	.7445	1.3432	.7720	1.2954	.8002	1.2497	.8292	1.2059	**20**
41	81	25	49	24	24	46	07	89	97	52	19
42	86	16	54	16	29	38	12	82	.8302	45	18
43	90	08	58	08	34	31	16	75	07	38	17
44	95	1.3899	63	00	38	23	21	67	12	31	16
45	99	91	67	1.3392	43	15	26	60	17	24	15
46	.7203	82	72	84	47	07	31	52	22	17	14
47	08	74	76	75	52	00	35	45	27	09	13
48	12	65	81	67	57	1.2892	40	37	32	02	12
49	17	57	85	59	61	84	45	30	37	1.1995	11
50	.7221	1.3848	.7490	1.3351	.7766	1.2876	.8050	1.2423	.8342	1.1988	**10**
51	26	40	95	43	71	69	55	15	46	81	9
52	30	31	99	35	75	61	59	08	51	74	8
53	34	23	.7504	27	80	53	64	01	56	67	7
54	39	14	08	19	85	46	69	1.2393	61	60	6
55	43	06	13	11	89	38	74	86	66	53	5
56	48	1.3798	17	03	94	30	79	78	71	46	4
57	52	89	22	1.3295	99	22	83	71	76	39	3
58	57	81	26	87	.7803	15	88	64	81	32	2
59	61	72	31	78	08	07	93	56	86	25	1
60	.7265	1.3764	.7536	1.3270	.7813	1.2799	.8098	1.2349	.8391	1.1918	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	69466	.71934	**60**
1	301	586	628	452	935	295	221	116	487	914	59
2	323	567	650	433	956	276	242	096	508	894	58
3	346	548	672	414	978	256	264	076	529	873	57
4	368	530	694	395	999	237	285	056	549	853	56
5	390	511	716	375	67021	217	306	036	570	833	55
6	412	492	738	356	043	198	327	016	591	813	54
7	435	473	759	337	064	178	349	72996	612	792	53
8	457	455	781	318	086	159	370	976	633	772	52
9	479	436	803	299	107	139	391	957	654	752	51
10	64501	76417	.65825	.75280	.67129	.74120	.68412	.72937	69675	.71732	**50**
11	524	398	847	261	151	100	434	917	696	711	49
12	546	380	869	241	172	080	455	897	717	691	48
13	568	361	891	222	194	061	476	877	737	671	47
14	590	342	913	203	215	041	497	857	758	650	46
15	612	323	935	184	237	022	518	837	779	630	45
16	635	304	956	165	258	002	539	817	800	610	44
17	657	286	978	146	280	.73983	561	797	821	590	43
18	679	267	.66000	126	301	963	582	777	842	569	42
19	701	248	022	107	323	944	603	757	862	549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	**40**
21	746	210	066	069	366	904	645	717	904	508	39
22	768	192	088	050	387	885	666	697	925	488	38
23	790	173	109	030	409	865	688	677	946	468	37
24	812	154	131	011	430	846	709	657	966	447	36
25	834	135	153	.74992	452	826	730	637	987	427	35
26	856	116	175	973	473	806	751	617	.70008	407	34
27	878	097	197	953	495	787	772	597	029	386	33
28	901	078	218	934	516	767	793	577	049	366	32
29	923	059	240	915	538	747	814	557	070	345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	**30**
31	967	022	284	876	580	708	857	517	112	305	29
32	989	003	306	857	602	688	878	497	132	284	28
33	.65011	.75984	327	838	623	669	899	477	153	264	27
34	033	965	349	818	645	649	920	457	174	243	26
35	055	946	371	799	666	629	941	437	195	223	25
36	077	927	393	780	688	610	962	417	215	203	24
37	100	908	414	760	709	590	983	397	236	182	23
38	122	889	436	741	730	570	.69004	377	257	162	22
39	144	870	458	722	752	551	025	357	277	141	21
40	.65166	.75851	.66480	74703	.67773	.73531	.69046	.72337	.70298	.71121	**20**
41	188	832	501	683	795	511	067	317	319	100	19
42	210	813	523	664	816	491	088	297	339	080	18
43	232	794	545	644	837	472	109	277	360	059	17
44	254	775	566	625	859	452	130	257	381	039	16
45	276	756	588	606	880	432	151	236	401	019	15
46	298	738	610	586	901	413	172	216	422	.70998	14
47	320	719	632	567	923	393	193	196	443	978	13
48	342	700	653	548	944	373	214	176	463	957	12
49	364	680	675	528	965	353	235	156	484	937	11
50	.65386	75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	**10**
51	408	642	718	489	.68008	314	277	116	525	896	9
52	430	623	740	470	029	294	298	095	546	875	8
53	452	604	762	451	051	274	319	075	567	855	7
54	474	585	783	431	072	254	340	055	587	834	6
55	496	566	805	412	093	234	361	035	608	813	5
56	518	547	827	392	115	215	382	015	628	793	4
57	540	528	848	373	136	195	403	.71995	649	772	3
58	562	509	870	353	157	175	424	974	670	752	2
59	584	490	891	334	179	155	445	955	690	731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	49°		48°		47°		46°		45°		′

′	40°		41°		42°		43°		44°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.8391	1.1918	.8693	1.1504	.9004	1.1106	.9325	1.0724	.9657	1.0355	60
1	96	10	98	1.1497	09	00	31	17	63	49	59
2	.8401	03	.8703	90	15	1.1093	36	11	68	43	58
3	06	1.1896	08	83	20	87	41	05	74	37	57
4	11	89	13	77	25	80	47	1.0699	79	31	56
5	16	82	18	70	30	74	52	92	85	25	55
6	21	75	24	63	36	67	58	86	91	19	54
7	26	68	29	56	41	61	63	80	96	13	53
8	31	61	34	50	46	54	69	74	.9702	07	52
9	36	54	39	43	52	48	74	68	08	01	51
10	.8441	1.1847	.8744	1.1436	.9057	1.1041	.9380	1.0661	.9713	1.0295	50
11	46	40	49	30	62	35	85	55	19	89	49
12	51	33	54	23	67	28	91	49	25	83	48
13	56	26	59	16	73	22	96	43	30	77	47
14	61	19	65	10	78	16	.9402	37	36	71	46
15	66	12	70	03	83	09	07	30	42	65	45
16	71	06	75	1.1396	89	03	13	24	47	59	44
17	76	1.1799	80	89	94	1.0996	18	18	53	53	43
18	81	92	85	83	99	90	24	12	59	47	42
19	86	85	90	76	.9105	83	29	06	64	41	41
20	.8491	1.1778	.8796	1.1369	.9110	1.0977	.9435	1.0599	.9770	1.0235	40
21	96	71	.8801	63	15	71	40	93	76	30	39
22	.8501	64	06	56	21	64	46	87	81	24	38
23	06	57	11	49	26	58	51	81	87	18	37
24	11	50	16	43	31	51	57	75	93	12	36
25	16	43	21	36	37	45	62	69	98	06	35
26	21	36	27	29	42	39	68	62	.9804	00	34
27	26	29	32	23	47	32	73	56	10	1.0194	33
28	31	22	37	16	53	26	79	50	16	88	32
29	36	15	42	10	58	19	84	44	21	82	31
30	.8541	1.1708	.8847	1.1303	.9163	1.0913	.9490	1.0538	.9827	1.0176	30
31	46	02	52	1.1296	69	07	95	32	33	70	29
32	51	1.1695	58	90	74	00	.9501	26	38	64	28
33	56	88	63	83	79	1.0894	06	19	44	58	27
34	61	81	68	76	85	88	12	13	50	52	26
35	66	74	73	70	90	81	17	07	56	47	25
36	71	67	78	63	95	75	23	01	61	41	24
37	76	60	84	57	.9201	69	28	1.0495	67	35	23
38	81	53	89	50	06	62	34	89	73	29	22
39	86	47	94	43	12	56	40	83	79	23	21
40	.8591	1.1640	.8899	1.1237	.9217	1.0850	.9545	1.0477	.9884	1.0117	20
41	96	33	.8904	30	22	43	51	70	90	11	19
42	.8601	26	10	24	28	37	56	64	96	05	18
43	06	19	15	17	33	31	62	58	.9902	1.0099	17
44	11	12	20	11	39	24	67	52	07	94	16
45	17	06	25	04	44	18	73	46	13	88	15
46	22	1.1599	31	1.1197	49	12	78	40	19	82	14
47	27	92	36	91	55	05	84	34	25	76	13
48	32	85	41	84	60	1.0799	90	28	30	70	12
49	37	78	46	78	66	93	95	22	36	64	11
50	.8642	1.1571	.8952	1.1171	.9271	1.0786	.9601	1.0416	.9942	1.0058	10
51	47	65	57	65	76	80	06	10	48	52	9
52	52	58	62	58	82	74	12	04	54	47	8
53	57	51	67	52	87	68	18	1.0398	59	41	7
54	62	44	72	45	93	61	23	92	65	35	6
55	67	38	78	39	98	55	29	85	71	29	5
56	72	31	83	32	.9303	49	34	79	77	23	4
57	78	24	88	26	09	42	40	73	83	17	3
58	83	17	94	19	14	36	46	67	88	12	2
59	88	10	99	13	20	30	51	61	94	06	1
60	.8693	1.1504	.9004	1.1106	.9325	1.0724	.9657	1.0355	1.0000	1.0000	0
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	49°		48°		47°		46°		45°		′

SOME IMPORTANT CONSTANTS AND THEIR LOGARITHMS

$\pi = 3.14159265$	$\log \pi = 0.497149873$
$2\pi = 6.28318531$	$\log 2\pi = 0.798179868$
$\frac{\pi}{2} = 1.57079633$	$\log \frac{\pi}{2} = 0.196119877$
$\frac{\pi}{4} = 0.78539816$	$\log \frac{\pi}{4} = 9.895089881 - 10$
$\pi^2 = 9.86960440$	$\log \pi^2 = 0.994299745$
$\sqrt{\pi} = 1.77245385$	$\log \sqrt{\pi} = 0.248574936$
$\sqrt{2\pi} = 2.50662827$	$\log \sqrt{2\pi} = 0.399089934$
$\frac{1}{\pi} = 0.318309886$	$\log \frac{1}{\pi} = 9.502850127 - 10$
$e = 2.718281829$	$\log e = 0.434294482$
$M = 0.434294482$	$\log M = 9.637784311 - 10$
$\frac{1}{M} = 2.302585092$	$\log \frac{1}{M} = 0.362215699$
1 radian = $57^\circ.2957795$	$\log 57.2957795 = 1.758122632$
1 radian = $3437'.74677$	$\log 3437.74677 = 3.536273883$
1 radian = $206264''.8062$	$\log 206264.8062 = 5.314425133$
1 degree = 0.017453293 radians	$\log 0.01745293 = 8.241877368 - 10$
1 minute = 0.000290888 radians	$\log 0.000290888 = 6.463726117 - 10$
1 second = 0.000004848 radians	$\log 0.000004848 = 4.685562611 - 10$